Taiwan International Mathematics Competition 2012 (TAIMC 2012)

World Conference on the Mathematically Gifted Students
---- the Role of Educators and Parents
Taipei, Taiwan, 23rd~28th July 2012

Invitational World Youth Mathematics Intercity Competition

Individual Contest

Section A. (5 points each)

Correct Answers:

1. **678**	2. $\frac{3\sqrt{3}}{2}$	3. **7161, 9361, 9812**	4. **4 cm^2**
5. **1458**	6. **672**	7. **4757**	8. **6 cm**
9. **160**	10. **536**	11. $\frac{2}{3}$	12. **196**

1. Determine the maximum value of the difference of two positive integers whose sum is 2034 and whose product is a multiple of 2034.

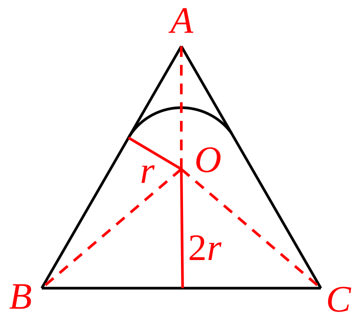
【Solution】

Let the two numbers be x and y. From $y = 2034 - x$, we have $xy = 2034x - x^2$. If this is divisible by 2034, then x^2 is divisible by 2034. Now 2034 = 2×32×113. Hence 2×3×113 = 678 must divide x, so that $678 \le x < 2034$. It follows that the only possible values for x are 678 and 2 × 678 = 1356. The corresponding values for y are 1356 and 678 respectively. Hence $x - y = \pm 678$ and its maximum value is 678.

ANS : 678

2. The diagram below shows a semicircle sitting on top of a square and tangent to two sides of an equilateral triangle whose base coincides with that of the square. If the length of each side of the equilateral triangle is 12 cm, what is the radius of the semicircle, in cm?

【Solution】



Let the triangle be ABC and let O be the centre of the semicircle. Let r be the radius of the semicircle. With the sides of the triangle as bases, the heights of triangles OAB, OAC and OBC are r, r and $2r$ respectively. Their total area is equal to the area of triangle ABC. Since the height of triangle ABC is $12 \times \frac{\sqrt{3}}{2}$, we have $4r = 6\sqrt{3}$ and $r = \frac{3\sqrt{3}}{2}$.

ANS : $\frac{3\sqrt{3}}{2}$

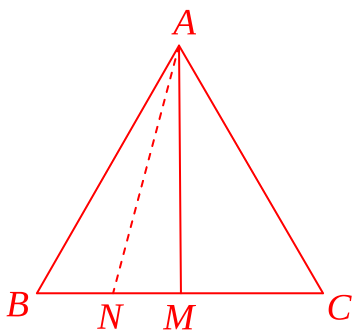
3. A four-digit number $\overline{abcd}$ is a multiple of 11, with $b + c = a$ and the two-digit number $\overline{bc}$ a square number. Find the number $\overline{abcd}$.

【Solution】

Since the two-digit number $\overline{bc}$ is a square number, and $b+c=a<10$, we have $\overline{bc}=16,25,36,81$. Since $\overline{abcd}$ is a multiple of 11, by trying possible digit d, we have $\overline{abcd}=7161,9361,9812$

ANS : 7161, 9361, 9812

4. The area of the equilateral triangle ABC is $8+4\sqrt{3}$ cm^2. M is the midpoint of BC. The bisector of $\angle MAB$ intersects BM at a point N. What is the area of triangle ABN, in cm^2?



【Solution】

We have $\frac{NM}{BN}=\frac{AM}{AB}=\frac{\sqrt{3}}{2}$. Hence $\frac{BM}{BN}=\frac{2+\sqrt{3}}{2}$ and $\frac{BC}{BN}=2+\sqrt{3}$. Denote the area of the triangle T by $[T]$.

Then $\frac{[ABC]}{[ABN]}=\frac{BC}{BN}=2+\sqrt{3}$. It follows that $[ABN]=\frac{[ABC]}{2+\sqrt{3}}=\frac{8+4\sqrt{3}}{2+\sqrt{3}}=4\text{ cm}^2$.

ANS : 4 cm^2

5. There is a 2×6 hole on a wall. It is to be filled in using 1×1 tiles which may be red, white or blue. No two tiles of the same colour may share a common side. Determine the number of all possible ways of filling the hole.

【Solution】

The top left space can be filled in 3 ways and the bottom left space can be filled in 2 ways, so that the first column from the left can be filled in 3×2=6 ways. In moving from column to column, we must retain at least one colour used in the preceding column. If we retain both colours, the only ways is to reverse the positions of the two tiles. If we retain just one colour, the tile with the repeated colour must be placed in a non-adjacent position, and the remaining space is filled with a tile of the third colour. Hence there are 3 ways to fill each subsequent column. It follows that the total number of ways is $6\times 3^5=1458$.

ANS : 1458

6. Let $N=1^9\times 2^8\times 3^7\times 4^6\times 5^5\times 6^4\times 7^3\times 8^2\times 9^1$. How many perfect squares divide N?

【Solution】

The prime factorization of N is $2^{30}\times 3^{13}\times 5^5\times 7^3$. Its largest square factor is $2^{30}\times 3^{12}\times 5^4\times 7^2$. Its square factors are the squares of the factors of $2^{15}\times 3^6\times 5^2\times 7^1$. Their number is (15+1)(6+1)(2+1)(1+1)=672.

ANS : 672

7. How many positive integers not greater than 20112012 use only the digits 0, 1 or 2?

【Solution】

The first few numbers are 1, 2, 10, 11, 12, 20, 21, 22, 100, 101 and so on. These are just numbers in base 3. The base 3 number 20112012 can be coverted to base 10 as follows.

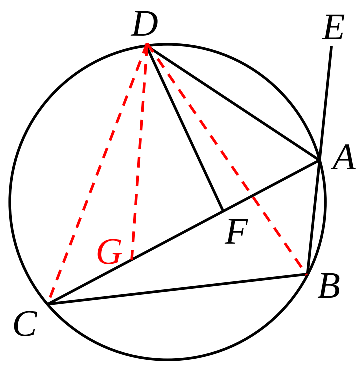
2	0	1	1	2	0	1	2
	+ 6	+ 18	+ 57	+ 174	+ 528	+ 1584	+ 4755
	6	19	58	176	528	1585	4757
× 3	× 3	× 3	× 3	× 3	× 3	× 3	
6	18	57	174	528	1584	4755	

Including the number 20112012 itself, there are 4757 positive integers which use only the digits 0, 1 and 2.

ANS：4757

8. The diagram below shows four points A, B, C and D on a circle. E is a point on the extension of BA and AD is the bisector of $\angle CAE$. F is the point on AC such that DF is perpendicular to AC. If $BA = AF = 2$ cm, determine the length of AC, in cm.

【Solution】



Let G be the point on AC such that $FG = AF = 2$ cm.
Then $GD = AD$ and $\angle DAG = \angle DGA$. Sine $ABCD$ is a cyclic quadrilateral, $\angle DCG = \angle DBA$. Moreover,

$$\begin{aligned}\angle DGC &= 180° - \angle DGA\\ &= 180° - \angle DAG\\ &= 180° - \angle DAE = \angle DAB.\end{aligned}$$

It follows that triangles DGC and DAB are congruent, so that $GC = BA = 2$ cm. Hence $AC = AD + DG + GC + 2 + 2 + 2 = 6$ cm.

ANS：6 cm

9. There are 256 different four-digit numbers $\overline{abcd}$ where each of a, b, c and d is 1, 2, 3 or 4. For how many of these numbers will $ad - bc$ be even?

【Solution】

Note that $ad - bc$ is even if ad and bc are either both odd or both even. The former occurs when all four numbers are odd. The number of this case is $2^4 = 16$. The latter occurs when a and d are not both odd, and b and c are not both odd. The number of this case is $(16 - 2^2)^2 = 144$. Hence there are $16 + 144 = 160$ possible numbers.

ANS：160

10. In a plane, given 24 evenly spaced points on a circle, how many equilateral triangles have at least two vertices among the given points?

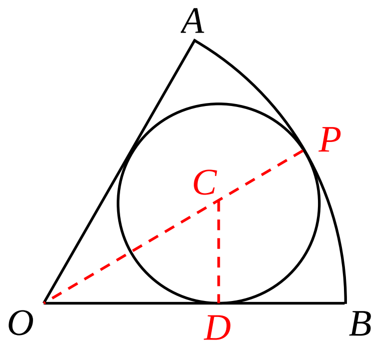
【Solution】

There are $\dfrac{24 \times 23}{2} = 276$ pairs of given points. For each pair, we can have an equilateral triangle on each side of the line joining them. However, some of these $2 \times 276 = 552$ triangles have been counted 3 times, because all three vertices are among the given points. There are 24 ÷ 3 = 8 such triangles. Hence the final count is 552 − 2 × 8 = 536.

ANS : 536

11. The diagram below shows a circular sector *OAB* which is one-sixth of a circle, and a circle which is tangent to *OA*, *OB* and the arc *AB*. What fraction of the area of the circular sector *OAB* is the area of this circle?

【Solution】



Let C be the centre of the circle and let the extension of OC cut arc AB at a point P. Let D be the point on OB such that CD is perpendicular to OB. Let $CD = r$. Then $OC = 2r$ and $CP = r$, so that $OP = 3r$. Hence the area of the sector is $\frac{1}{6}\pi(3r)^2 = \frac{3}{2}\pi r^2$ while the area of the circle is πr^2. The desired fraction is $\frac{2}{3}$.

ANS : $\frac{2}{3}$

12. An 8 × 8 chessboard is hung up on the wall as a target, and three identical darts are thrown in its direction. In how many different ways can each dart hit a different square such that any two of these three squares share at least one common vertex?

【Solution】

There are 7 pairs of adjacent rows and 7 pairs of adjacent columns, so that the number of 2 × 2 subboards is 7 × 7 = 49. The three darts must all hit a different square of some 2 × 2 subboard, and the square they miss can be any of the 4 squares in the subboard. Hence the total number of ways is 4 × 49 = 196.

ANS : 196

Section B. (20 points each)

1. What is the integral part of M, if

$$M = \sqrt{2012\times\sqrt{2013\times\sqrt{2014\times\sqrt{\cdots\sqrt{(2012^2-3)\times\sqrt{(2012^2-2)\times\sqrt{(2012^2-1)\times\sqrt{2012^2}}}}}}}}$$

【Solution】

The required answer is 2012.

By using the inequality $\sqrt{(N-1)(N+1)} < N$, we arrived that

$$\sqrt{(2012^2-1)\times\sqrt{2012^2}} < \sqrt{(2012^2-1)(2012^2+1)} < 2012^2,$$

It follows that

$$\sqrt{(2012^2-2)\times\sqrt{(2012^2-1)\times\sqrt{2012^2}}} < \sqrt{(2012^2-2)\times(2012^2)} < 2012^2-1,$$

$$\sqrt{(2012^2-3)\times\sqrt{(2012^2-2)\times\sqrt{(2012^2-1)\times\sqrt{2012^2}}}}<\sqrt{(2012^2-3)\times(2012^2-1)}$$
$$<2012^2-2$$

Repeating the same process, we conclude

$$M=\sqrt{2012\times\sqrt{2013\times\sqrt{2014\times\sqrt{\cdots\sqrt{(2012^2-2)\times\sqrt{(2012^2-1)\times\sqrt{2012^2}}}}}}}$$
$$<\sqrt{2012\times2014}$$
$$<2013$$

This implies the integral part of M is less than 2013.

And Conversely,

$\sqrt{2012^2}\geq2012$,

$\sqrt{(2012^2-1)\times\sqrt{2012^2}}>\sqrt{2012\times2012}=2012$,

$\sqrt{(2012^2-2)\times\sqrt{(2012^2-1)\times\sqrt{2012^2}}}>\sqrt{2012\times2012}=2012$,

$\sqrt{(2012^2-3)\times\sqrt{(2012^2-2)\times\sqrt{(2012^2-1)\times\sqrt{2012^2}}}}>\sqrt{2012\times2012}=2012$,

Continuing the same process, we have

$$M=\sqrt{2012\times\sqrt{2013\times\sqrt{2014\times\sqrt{\cdots\sqrt{(2012^2-2)\times\sqrt{(2012^2-1)\times\sqrt{2012^2}}}}}}}$$
$$>\sqrt{2012\times2012}$$
$$=2012$$

In summary, the integral part of M is 2012.

ANS：2012

【Marking Scheme】

- .... Showing M is at least 2012.. 5 points
- .... Showing M is less than 2013 .... 13 points
- .... Correct answer .... 2 points

2. Let m and n be positive integers such that

$$n^2<8m<n^2+60(\sqrt{n+1}-\sqrt{n})$$

Determine the maximum possible value of n.

【Solution】

When divided by 8, n^2 leaves a remainder no greater than 4.

Hence if $60(\sqrt{n+1}-\sqrt{n})<4$, then there will not be a multiple of 8 between n^2 and $n^2+60(\sqrt{n+1}-\sqrt{n})$. It follows that we must have $60(\sqrt{n+1}-\sqrt{n})\geq4$.

Hence $15 \geq \frac{1}{\sqrt{n+1}-\sqrt{n}} = \sqrt{n+1}+\sqrt{n} > 2\sqrt{n}$, so that $n \leq 56$.

When n = 55 or 56, $60(\sqrt{n+1}-\sqrt{n}) < 5$, and the remainder when 55^2 or 56^2 is divided by 8 is no greater than 1. Hence the desired multiple of 8 cannot exist either.

For n = 54, $60(\sqrt{55}-\sqrt{54}) = \frac{60}{\sqrt{55}+\sqrt{54}} \geq \frac{30}{\sqrt{55}} > 4$ since 30 × 30 = 900 > 880 = 4 × 4 × 55. Now 54^2 = 2916 so that $54^2 + 60(\sqrt{55}-\sqrt{54})$ > 2920. Since 2920 ÷8 = 365, we can take m = 365. It follows that the maximum value of n we seek is 54.

ANS : 54

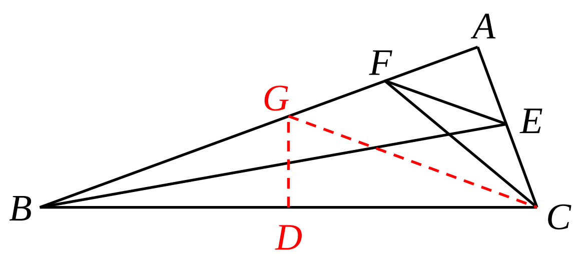
【**Marking Scheme**】

- .... Show that $60(\sqrt{n+1}-\sqrt{n}) \geq 4$ .... 11 points
- .... Solve the inequality .... 7 points
- .... Correct answer .... 2 points

3. Let ABC be a triangle with $\angle A = 90°$ and $\angle B = 20°$. Let E and F be points on AC and AB respectively such that $\angle ABE = 10°$ and $\angle ACF = 30°$. Determine $\angle CFE$.

【**Solution**】

Note that FC = 2AF. Let D be the midpoint of BC and let G be the point on AB such that GD is perpendicular to BC. Then triangles ABC and DBG are similar, so that $\frac{BD}{BG} = \frac{BA}{BC}$. By symmetry,



$\angle GCB = \angle GBC = 20°$, so that $\angle GCF = 20°$ also. Hence CG bisects $\angle BCF$ so that $\frac{FC}{FG} = \frac{BC}{BG}$. Since BE bisects $\angle ABC$, $\frac{BA}{BC} = \frac{AE}{CE}$. Now

$$\frac{AF}{FG} = \frac{\frac{1}{2}FC}{FG} = \frac{\frac{1}{2}BC}{BG} = \frac{BD}{BG} = \frac{BA}{BC} = \frac{AE}{EC}.$$

It follows that CG is parallel to EF, so that $\angle CFE = \angle GCF = 20°$.

ANS : $20°$

【**Marking Scheme**】

- .... Draw the correct auxilary line CG .... 3 points
- .... List equations of ratios of lengths .... 6 points
- .... Prove that CG is parallel to EF .... 9 points

- .... Correct answer .... 2 points