TEAM CONTEST

1. A positive real number is given. In each move, we can do one of the following: add 3 to it, subtract 3 from it, multiply it by 3 and divide it by 3. Determine all the numbers such that after exactly three moves, the original number comes back.

【Solution】

The operations of adding 3 and subtracting 3 are inverses of each other, as are the operations of multiplying by 3 and dividing by 3. If the same number is obtained after three operations, we can also achieve the same result by performing the inverses of these operations in reverse order. Sine the number of operations is odd, we cannot perform only additions and subtractions, nor can we perform only multiplications and divisions. Let the given number be x. We consider two cases.

Case I: Only one operation is multiplication or division.
By symmetry, we may assume that this operation is multiplication. There are three subcases.

Subcase I(a). The multiplication is the first operation.
The last two must both be subtractions. From $3x-3-3=x$, we have $x=3$.

Subcase I(b). The multiplication is the second operation.
If the first operation is addition, then the third operation cannot bring the number back to x. Hence after two operations, we have $3(x-3)$. If the third operation is addition, we have $3(x-3)+3=x$ and we get $x=3$ again. If the third operation is subtraction, we have $3(x-3)-3=x$ so that $x=6$.

Subcase I(c). The multiplication is the third operation.
The first two operations must both be subtractions. From $3(x-3-3)=x$, we have $x=9$.

Case II. Only one operation is additon or subtraction.
By symmetry, we may assume that this operation is subtraction. There are three subcases.

Subcase II(a). The subtraction is the first operation.
The last two operations must both be multiplications. From $3(3(x-3))=x$, we have $x=\frac{27}{8}$.

Subcase II(b). The subtraction is the second operation.
If the first operation is division, then the third operation cannot bring the number back to x. Hence after two operations, we have $3x-3$. The third operation must also be multiplicaiton. From $3(3x - 3) = x$, we have $x = \frac{9}{8}$.

Subcase II(c). The subtraction is the third operation.
The first two operations must both be multiplications. From $3(3x) - 3 = x$, we have $x = \frac{3}{8}$.

In summary, the possible values are $\frac{3}{8}$, $\frac{9}{8}$, $\frac{27}{8}$, 3, 6 and 9.

ANS : $\frac{3}{8}$, $\frac{9}{8}$, $\frac{27}{8}$, 3, 6 and 9

【**Marking Scheme**】

- Let k be the count of wrong/missing answers, $\text{Score} = 40 - \left\lceil \frac{20k}{3} \right\rceil$.

  (If the contestant missed the condition "positive", and get answers $0, -3, -6, -9, -\frac{3}{8}, -\frac{9}{8}, -\frac{27}{8}$, they only counts as one wrong answer.)

2. The average age of eight people is 15. The age of each is a prime number. There are more 19 year old among them than any other age. If they are lined up in order of age, the average age of the two in the middle of the line is 11. What is the maximum age of the oldest person among the eight?

【**Solution**】

Note that the total age of the eight people is $8 \times 15 = 120$. The sum of the ages of the two people in the middle of the line is 22. There are only three ways of expressing 22 as a sum of two prime numbers.

Case I. 11 and 11.
Because 19 appears more often than 11, the oldest three must all be 19, so that none of the youngest three can be 11. Their ages must add up to $120-2\times11+3\times19 = 41$, but the sum of three prime numbers less than 11 is at most $7+7+7=21$. This case is impossible.

Case II. 5 and 17.
The ages of the youngest four must be among 2, 3 and 5. By the Pigeonhole Principle, two of them are of the same age. Hence the oldest three must all be 19, and we have the same contradiction as in Case I.

Case III. 3 and 19.
The the oldest three cannot all be 19. Hence there are at most three people who are 19. Now the ages of the youngest four must be among 2 and 3, so that exactly two of them are 2 and the other two are 3. The age of the oldest person is therefore

120−2×2−2×3−3×19 = 53, which happens to be a prime number.

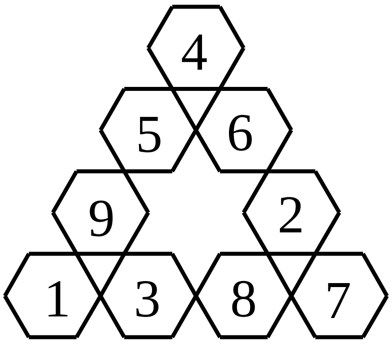
Thus the only possible age of the oldest person is 53.

ANS : 53

【Marking Scheme】

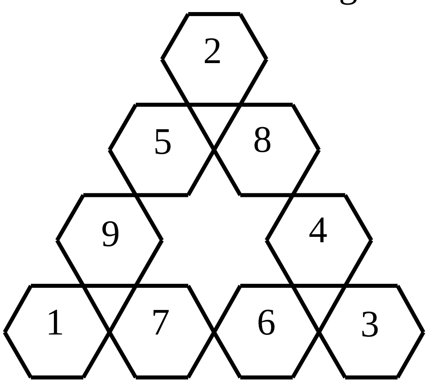
- .... Write out all possible case of the age of middle two people .... 4 points
- .... Case I .... 8 points
- .... Case II .... 12 points
- .... Case III .... 12 points
- .... Correct answer .... 4 points

3. In the diagram below, the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9 are placed one inside each hexagon, so that the sum of the numbers inside the four hexagons on each of the three sides of the triangle is 19. If you are allowed to rearrange the numbers but still have the same sum on each side, what is the smallest possible sum and what is the largest possible sum?

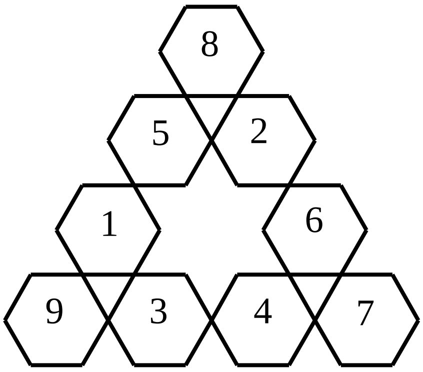


【Solution】

Each number is counted once except for the three at the corners of the triangle.
To minimize the constant sum, we put 1, 2 and 3 there. Since 1+4+7−1−2−3 = 6 and 6÷ 3 = 2, the minimum sum is 19 −2 = 17. The diagram below shows that the value may be attained.



To maximize the constant sum, we put 7, 8 and 9 there. Since 7+8+9−1−4−7 = 12 and 12÷3 = 4, the maximum sum is 19+4=23. The diagram below shows that the value may be attained.



ANS : The minimum sum is 17 and the maximum sum is 23

【**Marking Scheme**】

- .... Minimum sum.. 20 points
- .... Maximum sum.. 20 points

4. There are 2012 evenly spaced points on a line. Each is to be painted orange or green. If three points *A*, *B* and *C* are such that *AB* = *BC*, then if *A* and *C* are colored by the same color, so is *B*. Determine the number of all possible ways of painting these points.

【**Solution**】

Let these points from left to right be $p_1, \cdots, p_{2012}$. WLOG assume that p_1 is green. If not all points are green, we consider the minimum x such that p_{x+1} is orange, then p_x is green. Since $\overline{p_x p_{x+1}} = \overline{p_{x+1} p_{x+2}}$, p_{x+2} must also be orange. We have the following lemma:

Lemma 1:

If p_x is green, p_{x+1} is orange, then for every $k > 0$ that satisfy $x + 2k \leq 2012$, points $p_{x+1}, \cdots, p_{x+2k}$ are all orange.

Proof of Lemma 1:

We prove it by induction on k.

When $k = 1$, We've already proved that both p_{x+1} and p_{x+2} are both orange.

If the lemma is correct for $k = k_0$, that is, all points $p_{x+1}, \cdots, p_{x+2k_0}$ are all orange.

When $k = k_0 + 1$, we have to prove that p_{x+2k_0+1} and p_{x+2k_0+2} are both orange.

Since $\overline{p_x p_{x+k_0+1}} = \overline{p_{x+k_0+1} p_{x+2k_0+2}}$, and p_x is green, $k_0 + 1 \leq 2k_0$, so by induction hypothesis, p_{x+k_0+1} is orange. So p_{x+2k_0+2} must be orange too. Since $\overline{p_{x+2k_0} p_{x+2k_0+1}} = \overline{p_{x+2k_0+1} p_{x+2k_0+2}}$, and both p_{x+2k_0} and p_{x+2k_0+2} are orange,

p_{x+2k_0+1} must be orange too. So the lemma also holds at $k = k_0 + 1$.

By induction, the lemma is proved.

Back to the original problem, we consider the following 2 cases:
Case I. $x = 1$, then by lemma 1, we'll have $p_2, \ldots, p_{2011}$ are all orange, and p_{2012} can be green or orange.
Case II. $x > 1$, then by lemma 1, we'll have $p_{x+1}, \ldots, p_{2011}$ are all orange, and since either the midpoint of $\overline{p_{x-1}p_{2012}}$ or $\overline{p_x p_{2012}}$ would be one of orange points, we must have p_{2012} is orange.

In conclusion, all possible solutions are of form OO…OGG…G, GG…GOO…O, OGG…GO, GOO…OG, where O stands for Orange, and G stands for Green. The first 2 cases both have 2012 possible solutions, so there are in total 2012×2+2 = 4026 solutions.

ANS：4026

【Marking Scheme】

- .... Prove lemma 1 or its equivalent .... 28 points
- .... Show solution of the form OO…OGG…G .... 4 points
- .... Show solution of the form OG…GO .... 4 points
- .... Correct answer .... 4 points

5. Consider the four-digit number 2012. We can divide it into two numbers in three ways, namely, 2|012, 20|12 and 201|2. If we multiply the two numbers in each pair and add the three products, we get 2×012+20×12+201×2=666. Find all other four-digit numbers which yield the answer 666 by this process.

【Solution】

Let the number of 1000*a*+100*b*+10*c*+*d*, where *a* is a non-zero digit while *b*, *c* and *d* are any digits. Then 100*ab*+110*ac*+111*ad*+10*bc*+11*bd*+*cd*=666. Note that $d \neq 0$ as otherwise 100*ab*+110*ac*+10*bc* $\not\equiv$ 6 (mod 10). We consider six cases.

Case I. *ad* = 6.
Then 111*ad* = 666 so that all other terms must be 0, which means *b*=*c*=0. Hence we have 1006, 2003, 3002 and 6001.

Case II. *ad* = 5.
Then we have either 511*b*+551*c*+10*bc* =111 or 155*b*+115*c*+10*bc* =111. We must also have *b* =*c* =0, but the equation is not satisfied.

Case III. *ad* = 4.
We have three subcases.

Subcase III(a). *a* = 4 and *d* = 1.

Then $411b+441c+10bc = 222$, which forces $b = c = 0$. The equation is not satisfied.

Subcase III(b). $a = 1$ and $d = 4$.
Then $144b+114c+10bc = 222$, which forces $b = 0$ or $c = 0$. The equation is not satisfied.

Subcase III(c). $a = d = 2$.
Then $222b+222c+10bc = 222$. We have either $b = 0$ and $c = 1$ or $b = 1$ and $c = 0$, yielding 2012 or 2102.

Case IV. $ad = 3$.
If $a = 3$ and $d = 1$, then $311b+331c+10bc = 333$. Hence one of b and c is 1 and the other is 0, but the equation is not satisfied. If $a = 1$ and $d = 3$, then $133b+113c+10bc = 333$ so that $b + c \equiv 1 \pmod{10}$. The equation cannot be satisfied.

Case V. $ad = 2$.
If $a = 2$ and $d = 1$, then $211b+221c+10bc = 444$ so that $b+c \equiv 4 \pmod{10}$. The equation cannot be satisfied. If $a=1$ and $d=2$, then $122b+112c+10bc = 444$ so that $b + c \equiv 2 \pmod 5$. The equation cannot be satisfied.

Case VI. $ad = 1$.
Then $a = d = 1$ and $111b+111c+10bc = 555$. This is only possible for $b = 0$ and $c = 5$ or $b = 5$ and $c = 0$, yielding 1051 and 1501.

In summary, apart from 2012, the other numbers with the desired property are 1006, 1051, 1501, 2003, 2102, 3002 and 6001.

ANS：1006, 1051, 1501, 2003, 2102, 3002 and 6001

【Marking Scheme】

● .... Each wrong/missing answer .... -6 points

6. Let n be a positive integer such that $2n$ has 8 positive factors and $3n$ has 12 positive factors. Determine all possible numbers of positive factors of $12n$.

【Solution】

Note that $8 = 7 + 1 = (3 + 1)(1 + 1) = (1 + 1)(1 + 1)(1 + 1)$. Since $2n$ has 8 positive factors, it is either the 7th power of a prime, the product of a prime and the cube of another prime, or the product of 3 different primes. We consider these cases separately.

Case I. $2n = p^7$ for some prime p.
We must have $p = 2$, but then $3n = 3\times 2^6$ has $(1 + 1)(6 + 1) = 14$ positive factors instead of 12. This is impossible.

Case II. $2n = p^3q$ where p and q are different primes.
Suppose $q = 2 < p$. If $p = 3$, then $3n = 3^4$ has $4+1=5$ positive factors. If $p > 3$, then $3n = 3q^3$ has $(1 + 1)(3 + 1) = 8$ positive factors. Neither is possible. Suppose $p = 2 < q$. If $q = 3$, then $3n = 2^2 \times 3^2$ has $(2 + 1)(2 + 1) = 9$ positive factors. This is also impossible. If $q > 3$, then $3n = 2^2 \times 3q$ has $(2 + 1)(1 + 1)(1 + 1) = 12$ positive factors, which satisfies the given condition. Hence $12n = 2^4 \times 3q$ has $(4 + 1)(1 + 1)(1 + 1) = 20$

positive factors.

Case III. $2n = pqr$ where p, q and r are different primes.
By symmetry, we may take $r = 2$ and $q < p$. If $q = 3$, then $3n = 3^3p$ has $(2 + 1)(1 + 1) = 6$ positive factors. If $q > 3$, Then $3n = 3pq$ has $(1+1)(1+1)(1+1) = 8$ positive factors. Neither is possible.

In summary, we must have $n = 48q$ for some prime $q > 3$, and it has 20 positive factors.

ANS : 20

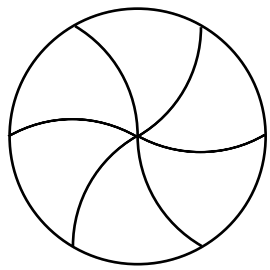
【**Marking Scheme**】

- .... Consider factorization and write out the relation with the number of divisors. .... 8 points
- .... Case I .... 8 points
- .... Case II .... 8 points
- .... Case III .... 8 points
- .... Existence of such n.. 4 points
- .... Correct answer .... 4 points

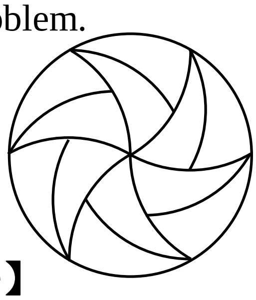
7. Use straight and circular cuts to dissect a circle into at least two congruent pieces. There must be at least one piece which does not contain the centre of the circle in its interior or on its perimeter.

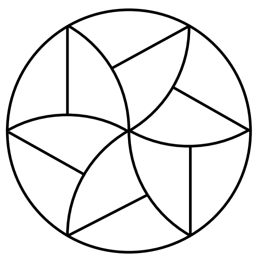
【**Solution**】

The diagram on the right shows a six-piece dissection in which every piece contains the centre of the circle on its perimeters. Thus it is not a solution. However, it is a good first step towards a solution.



The diagram below shows the second step of two different twelve-piece dissections which satisfy the problem.





【**Marking Scheme**】

- .... Correct answer .... 40 points

8. A machine consists of three boxes each with a red light that is initially off. When objects are put into the boxes, the machine checks the total weight in each box. If the total weight in one box is strictly less than the total weight in each of the other two boxes, the red light of that box will go on. Otherwise, all red lights remain off. Use this machine twice to find a fake ball among seven balls which is heavier than the other six. The other six are of equal weight.

【Solution】

Label the balls 1 to 7. In the first weighing, put balls 1 and 2 in the first box, balls 3 and 4 in the second box and balls 5, 6 and 7 into the third box. The red light of the third box cannot go on. There are three cases.

Case I. No red lights go on.
Then one of ball 5, 6 and 7 is heavy. In the second weighing, put ball 5 in the first box, ball 6 is in the second box and put two of the other five balls in the third box. Again, the red light of the third box cannot go on. If no red lights go on, then ball 7 is heavy. If the red light of the first box goes on, then ball 6 is heavy. If the red light on the second box goes on, then ball 5 is heavy.

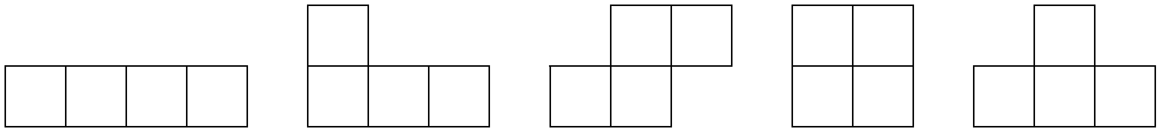
Case II. The red light of the first box goes on.
Then one of balls 3 and 4 is heavy. In the second weighing, put ball 3 in the first box, ball 4 is in the second box and put two of the other five balls in the third box. The red light of either the first box or the second box must go on, and the heavy ball can be found.

Case III. The red light of the second box goes on.
This is analogous to Case II, with one of balls 1 and 2 being heavy.

【Marking Scheme】

- .... Consider put 2, 2, 3 balls into each box and explain the result .... 24 points
- .... Consider put 1, 1, 2 balls into each box and explain the result .... 16 points

9. The diagram below shows all five pieces which can be formed of four unit squares joined edge to edge. They are called the I-, L-, N-, O- and T-Tetrominoes.



(a) Use three different pieces to construct a figure with reflectional symmetry. Find five solutions.

(b) Use three different pieces to construct a figure with rotational symmetry. Find one solution.

【**Solution**】

(a) Five constructions are shown in the diagram below, using the combinations TIN, LON, LIT, LNT and NIL For some combinations, there are other constructions

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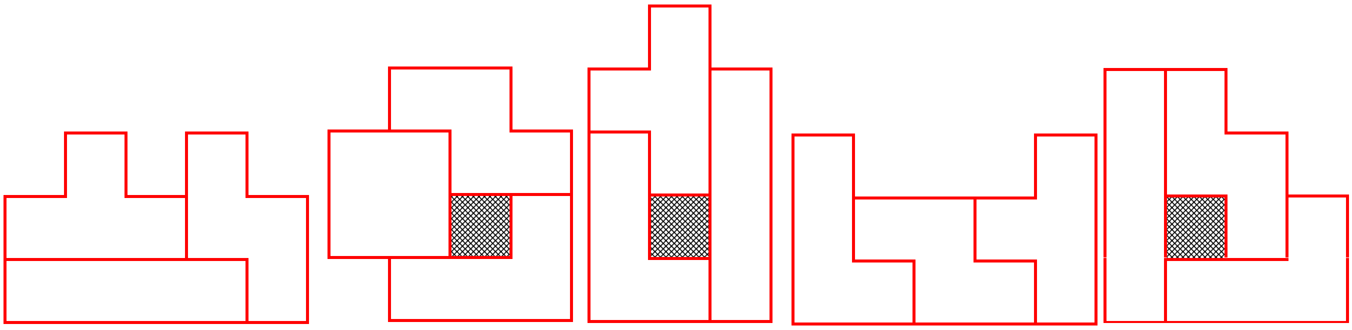
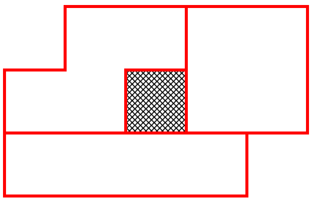


(b) A solution is shown in the diagram below, using the combination ION.



【**Marking Scheme**】

● .... Correct answer of (a)
....3 points each(15 max)

● .... Correct answer of (b)
....25 points

10. The digits in base 10 have been replaced in some order by the letters *A*, *B*, *C*, *D*, *E*, *F*, *G*, *H*, *I* and *J*. We have three clues.

(1) The two-digit number *AB* is the product of *A*, *A* and *C*.

(2) The two-digit number *DE* is the product of *C* and *F*.

(3) The two-digit number *BG* is the sum of *H*, *I* and the product of *F* and *G*.

What digit is replaced by the letter *J* ?

【**Solution**】

Note that none of *A*, *B* and *D* can be 0. We have

$$10A+B=A^2C \qquad (1)$$

$$10D+E=CF \qquad (2)$$

$$10B+G=H+I+FG \qquad (3)$$

Consider (1). If $A = 1$, $10\le 10+B = C\le 9$, which is a contradiction. If A=2. $20+B = 4C$. We have B =4 and C6, or $B = 8$ and C =7. If A=3, $30+B$ =9C. We must have B =6 and

C =4. If A = 4, 40 +B =16C. We must have B =8 and C = 3. Suppose B = 8. Then $H + I \leq 7+9 = 16$ and $FG \leq 7 \times 9 = 63$. By (3), $80 \leq 80+G = H + I + FG \leq 16+63 = 79$, which is a contradiction. We now have two cases.

Case 1. B = 4.
Then A = 2 and C = 6. Now (2) becomes $10D + E = 6F$. Hence E is even. We have two subcases.

Subcase 1(a). E = 0.
From (2), F = 5 and D = 3, so that only the digits 1, 7, 8 and 9 are left. Now (3) becomes $40 = H + I + 4G$. The only solution here is G = 8 and $\{H, I\} = \{1, 7\}$.

Subcase 1(b). E = 8.
From (2), F = 3 and D = 1, so that only the digits 0, 5, 7 and 9 are left. Now (3) becomes $40 = H + I + 2G < 40$, which is a contradiction.

Case 2. B = 6.
Then A = 3 and C = 4. Now (2) becomes $10D + E = 4F$. Again E is even. We have three subcases.

Subcase 2(a). E = 0.
From (2), F = 5 and D = 2, so that only the digits 1, 7, 8 and 9 are left. Now (3) becomes $60 = H + I + 4G < 60$, which is a contradiction.

Subcase 2(b). E = 2.
From (2), we have either F = 3 and D = 1 or F = 8 and D = 3. Neither is possible as we already have A = 3.

Subcase 2(c). E = 8.
From (2), we have two possibilities. If F = 2 and D = 1, then (3) becomes $60 = H+I+G < 30$, which is a contradiction. Suppose F = 7 and D = 2, so that only the digits 1, 5, 8 and 9 are left. Now (3) becomes $60 = H + I + 6G$. There is a solution G = 9 and $\{H, I\} = \{1, 5\}$.

In conclusion, there are two solutions:

$A = 2, B = 4, C = 6, D = 3, E = 0, F = 5, G = 8, \{H, I\} = \{1, 7\}$ and $J = 9$.

$A = 3, B = 6, C = 4, D = 2, E = 8, F = 7, G = 9, \{H, I\} = \{1, 5\}$ and $J = 0$.

ANS : 0, 9

【Marking Scheme】
- ● List all case of (A, B, C) .... 8 points
- ● Prove cases for B=8 is impossible .... 8 points
- ● Case 1 and solution 1 .... 12 points
- ● Case 2 and solution 2 .... 12 points