Taiwan International Mathematics Competition 2012 (TAIMC 2012)

World Conference on the Mathematically Gifted Students
---- the Role of Educators and Parents
Taipei, Taiwan, 23rd~28th July 2012

Elementary Mathematics International Contest

Individual Contest

1. In how many ways can 20 identical pencils be distributed among three girls so that each gets at least 1 pencil?

【Solution】

The first girl can take from 1 to 18 pencils. If she takes 1, the second girl can take from 1 to 18. If she takes 2, the second girl can take from 1 to 17, and so on. The third girl simply takes whatever is left. Hence the total number of ways is

$18+17+\ldots+1=\dfrac{19\times 18}{2}=171.$

ANS: 171

2. On a circular highway, one has to pay toll charges at three places. In clockwise order, they are a bridge which costs \$1 to cross, a tunnel which costs \$3 to pass through, and the dam of a reservoir which costs \$5 to go on top. Starting on the highway between the dam and the bridge, a car goes clockwise and pays toll-charges until the total bill amounts to \$130. How much does it have to pay at the next place?

【Solution】

Since 1+3+5=9, it takes \$9 to go once around the highway. When 130 is divided by 9, the quotient is 14, but more importantly the remainder is 4. This means that after completing a number of round which happens to be 14, the car has pay an extra \$4, which means it has crossed the bridge and passed through the tunnel. At the next place, it will have to pay \$5 to go on top of the dam of the reservoir.

ANS: \$5

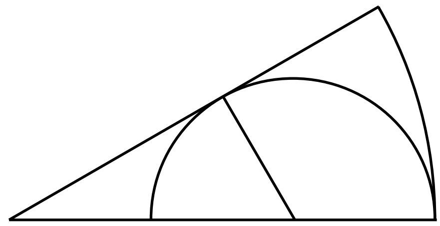
3. When a two-digit number is increased by 4, the sum of its digits is equal to half of the sum of the digits of the original number. How many possible values are there for such a two-digit number?

【Solution】

Clearly, a carrying occurs when the number is increased by 4. Hence its units digit is 6, 7, 8 or 9. Suppose it is 6. With this 6 turning into a 1 in the tens dight, there is a net loss of 5. Hence the sum of the digits of the original number must be 2×5=10, so that it is 46. Indeed, the sum of the digits of 46 is 10 and the sum of the digits of 46+4=50 is 5, which is half of 10. Using the same reasoning, we see that if the units digit is 7, 8 or 9, the original number must be 37, 28 and 19. Hence there are 4 possible values.

ANS: 4

4. In the diagram below, OAB is a circular sector with $OA = OB$ and $\angle AOB = 30°$. A semicircle passing through A is drawn with centre C on OA, touching OB at some point T. What is the ratio of the area of the semicircle to the area of the circular sector?

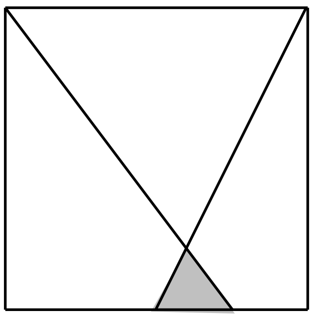


【Solution】

We may take the length of $OA = OB$ to be 6 cm. The area of a circle with radius 6 cm is $\pi \times 6^2 = 36\pi\ \text{cm}^2$, so that the area of a $30°$ sector is $36\pi \times \frac{30°}{360°} = 3\pi\ \text{cm}^2$. Note that COT is half an equilalteral triangle. Hence OC=2CT=2CA so that OA=3CT. Since OA=6, CT=2 and the area of the semicircle is $\frac{1}{2}\pi \times 2^2 = 2\pi\ \text{cm}^2$. The desired ratio is therefore 2:3.

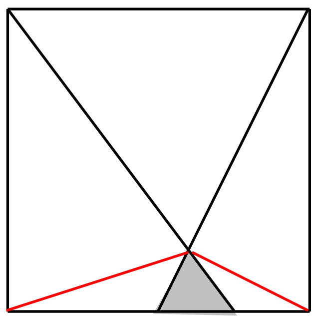
ANS: 2:3

5. *ABCD* is a square with total area 36 cm^2. F is the midpoint of AD and E is the midpoint of FD. BE and CF intersect at G. What is the area, in cm^2, of triangle EFG?



【Solution】

The area of triangle *GED* is equal to that of triangle *EFG* because *ED*=*EF*. The area of triangle *GAF* is twice that of triangle *EFG* because *AF*=2*FE*. The area of triangle *GBC* is sixteen times that of triangle *EFG* because the two triangles are similar and *BC*=4*EF*. Hence the area of triangle *EFG* is $\frac{1}{1+1+2+16} = \frac{1}{20}$ that of the total area of triangles *ADG* and *BCG*, which is half that of the square *ABCD*. It follows that the area of triangle *EFG* is $\frac{1}{40}$ that of the square *ABCD*, so the area of triangle *EFG* is 0.9.



ANS: 0.9

6. In a village, friendship among girls is mutual. Each girl has either exactly one friend or exactly two friends among themselves. One morning, all girls with two friends wear red hats and the other girls all wear blue hats. It turns out that any two friends wear hats of different colours. In the afternoon, 10 girls change their

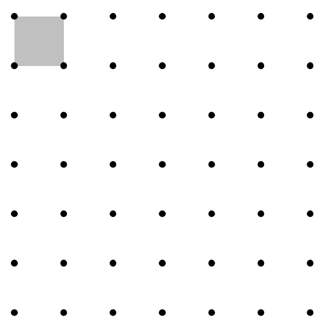
red hats into blue hats and 12 girls change their blue hats into red hats. Now it turns out that any two friends wear hats of the same colour. How many girls are there in the village?

【Solution】

Consider a girl with only one friend. She will be wearing a blue hat in the morning, and her only friend will be wearing a red hat. This friend has a second friend who must be wearing a blue hat, and the friendship network stops there. It follows that the girls in the village may be divided into groups of three, with one being friends with the other two, but the other two are not friends of each other. In the afternoon, in order for any two friends to be wearing hats of the same colour, all three girls in each group must wear hats of the same colour. This can be accomplished by either the girl in red changing her hat, or the two girls both changing their hats. It follows that in 10 groups, the girl in red hat changes, and in $12 \div 2 = 6$ groups, the girls in blue hats change. The total number of groups is 10+6=16 and the total number of girls is $16 \times 3 = 48$.

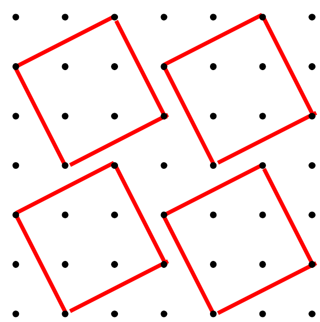
ANS: 48

7. The diagram below shows a 7×7 grid in which the area of each unit cell (one of which is shaded) is 1 cm^2. Four congruent squares are drawn on this grid. The vertices of each square are chosen among the 49 dots, and two squares may not have any point in common. What is the maximum area, in cm^2, of each of these four squares?



【Solution】

Because the vertices of the squares are chosen among the dots, the side-length of each square, in cm, is either a positive integer, or by Pythagoras' Theorem, the square root of the sum of two positive integers. In either case, its area in cm^2 is a positive integer. Since the area of the whole grid is 36 cm^2, this integer is at most 9 cm^2, and cannot be 6 or 7 cm^2. It cannot be 9 cm^2, as otherwise the squares will have boundary points in common. It cannot be 8 cm^2 as we cannot fit even two copies of such a square without them having common points. Hence the maximum area of each square is 5 cm^2, and the diagram below shows that this can be attained.



ANS: 5

8. The sum of 1006 different positive integers is 1019057. If none of them is greater than 2012, what is the minimum number of these integers which must be odd?

【Solution】

Suppose we take the first 1006 even numbers. Then their sum is $\frac{1006\times(2+2012)}{2} =$ 1013042. Now $1019057 - 1013042 = 6015$. So we have to trade some even numbers for an equal amount of odd numbers. Clearly, trading only one number can raise the total by at most 2011−2=2009. Hence we have to trade more than one number. However, if we trade exactly two numbers, the total will increase by an even amount, which is not what we want. Hence we must trade at least three numbers. If we trade 2, 4 and 6 for 2007, 2009 and 2011, the total will increase by 2007+2009+2011−2−4−6 =3×(2009−4)=6015 exactly. Hence the minimum number of the 1006 integers which must be odd is 3.

ANS: 3

9. The desks in the TAIMC contest room are arranged in a 6 × 6 configuration. Two contestants are neighbours if they occupy adjacent seats along a row, a column or a diagonal. Thus a contestant in a seat at a corner of the room has 3 neighbours, a contestant in a seat on an edge of the room has 5 neighbours, and a contestant in a seat in the interior of the room has 8 neighbours. After the contest, a contestant gets a prize if at most one neighbour has a score greater than or equal to the score of the contestant. What is maximum number of prize-winners?

【Solution】

Divide the contest room into 9 sections each of which is a 2 × 2 configuration. In each section, arrange the four contestants in order of their scores starting with the highest. Then the contestant third or fourth in the line cannot be prize-winners because each would have at least two neighbours whose scores are not lower. Hence the maximum number of prize-winners is 2 × 9 = 18. The diagram below shows the marks of the contestants in the respective seats, and each contestant in the first, third or fifth row get a prize.

40	50	60	70	80	90
10	10	10	10	10	10
40	50	60	70	80	90
10	10	10	10	10	10
40	50	60	70	80	90
10	10	10	10	10	10

ANS: 18

10. The sum of two positive integers is 7 times their difference. The product of the same two numbers is 36 times their difference. What is the larger one of these two numbers?

【Solution】

Suppose the difference is 1. Then the sum is 7. If we add twice the smaller number to the difference, we will get the sum. Hence the smaller number is 3, and the larger number is 4. Now the product is 12, which is 12 times the difference. However, it is given that the product is 36 times the difference. Since 36 ÷ 12 = 3, the smaller number is 3 × 3 = 9 and the larger number is 3 × 4 = 12.

ANS: 12

11. In a competition, every student from school A and from school B is a gold medalist, a silver medalist or a bronze medalist. The number of gold medalist from each school is the same. The ratio of the percentage of students who are

gold medalist from school A to that from school B is 5:6. The ratio of the number of silver medalists from school A to that from school B is 9:2. The percentage of students who are silver medalists from both school is 20%. If 50% of the students from school A are bronze medalists, what percentage of the students from school B are gold medalists?

【Solution】

Suppose school A has 9 silver medalists. Then school B has 2, for a total of 9+2=11. Hence the total population of the two schools is 11 ÷ 20% = 55. For gold medalists, the numbers are the same but the percentages are in the ratio 5:6. This means that the ratio of the populations are in the ratio 6:5. It follows that the population of school A is $55\times\frac{6}{6+5}=30$ and that of school B is 55−30 = 25. Now the number of bronze medalists from school A is 30 × 50% = 15, so that the number of gold medalists from each school is 30−15−9=6. It follows that the percentage of students from school B who are gold medalists is 6 ÷ 25 = 24%.

ANS: 24%

12. We start with the fraction $\frac{5}{6}$. In each move, we can either increase the numerator by 6 or increases the denominator by 5, but not both. What is the minimum number of moves to make the value of the fraction equal to $\frac{5}{6}$ again?

【Solution】

Since the value of the fraction is unchanged, the ratio of the amount added to the numerator and the amount added to the denominator is also 5:6. Since we are adding 6s to the numerator and 5s to the denominator, the number of 6s added must be a multiple of 25 and the number of 5s added must be the same multiple of 36. Hence the minimum number of moves is 25+36= 61.

ANS: 61

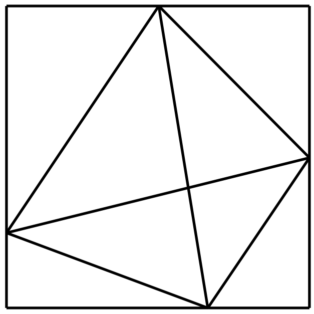
13. Five consecutive two-digit numbers are such that 37 is a divisor of the sum of three of them, and 71 is also a divisor of the sum of three of them. What is the largest of these five numbers?

【Solution】

Among five consecutive numbers, the sum of the largest three is only 6 more than the sum of the smallest three. Thus we are looking for a multiple of 37 and a multiple of 71 which differ by at most 6. Now 37 × 2 − 71 = 3 and 37 × 4 − 71 × 2 = 6. In the latter case, 148 must be the sum of the largest three numbers, which are consecutive. Hence the sum is divisible by 3, but 148 is not a multiple of 3. In the former case, note that 71 < 23 + 24 + 25 = 72 < 74. Hence the smallest of the five numbers cannot be 23 or more, and the largest cannot be 25 or less. This means that the five numbers must be 22, 23, 24, 25 and 26, the largest of which is 26. There is no need to consider higher multiples of 37 and 71 as the sums will be too large to allow the five numbers to have only two digits.

ANS: 26

14. *ABCD* is a square. *M* is the midpoint of *AB* and *N* is the midpoint of *BC*. *P* is a point on *CD* such that *CP* = 4 cm and *PD* = 8 cm, *Q* is a point on *DA* such that *DQ* = 3 cm. *O* is the point of intersection of *MP* and *NQ*. Compare the areas of the two triangles in each of the pairs (*QOM*, *QAM*), (*MON*, *MBN*), (*NOP*, *NCP*) and (*POQ*, *PDQ*). In cm^2, what is the maximum value of these four differences?



【Solution】

We have *AM* =*MB* =*BN* =*NC* =6 cm and *AQ* = 9 cm. It follows that the area of *ABNQ* is $\frac{1}{2}\times12\times(9+6)$ = 90 cm^2. Hence the area of *QMN* is $90-\frac{1}{2}\times9\times6-\frac{1}{2}\times6\times6=$ 45 cm^2. The area of *QNCD* is 12×12−90 = 54 cm^2. Hence the area of *NPQ* is $54-\frac{1}{2}\times3\times8-\frac{1}{2}\times6\times4=$ 30 cm^2. Similarly, the area of *PQM* is 45 cm^2 and the area of *MNP* is 30 cm^2. It follows that *QO* : *ON* = 45 : 30 = 3 : 2. Hence the area of *QOM* is $45\times\frac{3}{3+2}=$ 27 cm^2, which is the same as the area of *QAM*. The area of *MON* is 45 − 27 = 18 cm^2, which is the same as the area of *MBN*. Similarly, the area of *NOP* is $30\times\frac{2}{3+2}=$ 12 cm^2, which is the same as the area of *NCP*. Finally, the area of *POQ* is 30 − 12 = 18 cm^2, which is 18 − 12 = 6 cm^2 more than the area of *PDQ*. The maximum value of the four difference is therefore 6 cm^2.

ANS: 6

15. Right before Carol was born, the age of Eric is equal to the sum of the ages of Alice, Ben and Debra, and the average age of the four was 19. In 2010, the age of Debra was 8 more than the sum of the ages of Ben and Carol, and the average age of the five was 35.2. In 2012, the average age of Ben, Carol, Debra and Eric is 39.5. What is the age of Ben in 2012?

【Solution】

Right before Carol was born, the total age of Alice, Ben, Debra and Eric was 19 × 4 = 76. In 2010, the total age of all five was 35.2 × 5 = 176. It follows that in 2010, the age of Carol was (176 − 76) ÷ 5 = 20, so that she was born in 1990. In 2012, the total age of Ben, Carol, Debra and Eric is 39.5 × 4 = 158. In 2010, the total age of these four was 158 − 2 × 4 = 150. It follows that the age of Alice was 176 − 150 = 26. Note that the age of Eric in 1990 was 76 ÷ 2 = 38. Hence the age of Eric in 2010 was 58, and the total age of Ben, Carol and Debra was 176 − 26 − 58 = 92. It follows that the age of Debra was (92 + 8) ÷ 2 = 50, and the age of Ben was 92 − 50 − 20 = 22. Thus the age of Ben in 2012 is 22 + 2 = 24.

ANS: 24