Taiwan International Mathematics Competition 2012 (TAIMC 2012)

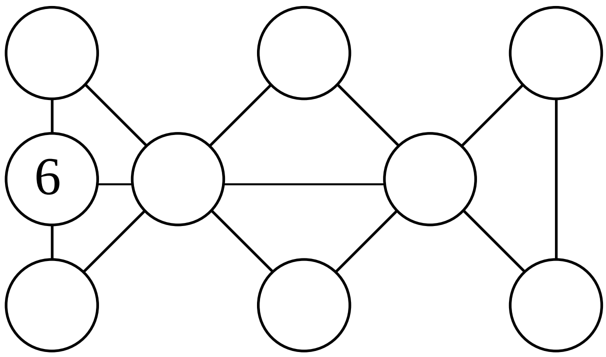
World Conference on the Mathematically Gifted Students
---- the Role of Educators and Parents
Taipei, Taiwan, 23rd~28th July 2012

Elementary Mathematics International Contest

TEAM CONTEST

(with solution and marking scheme)

1. Each of the nine circles in the diagram below contains a different positive integer. These integers are consecutive and the sum of numbers in all the circles on each of the seven lines is 23. The number in the circle at the top right corner is less than the number in the circle at the bottom right corner. Eight of the numbers have been erased. Restore them.

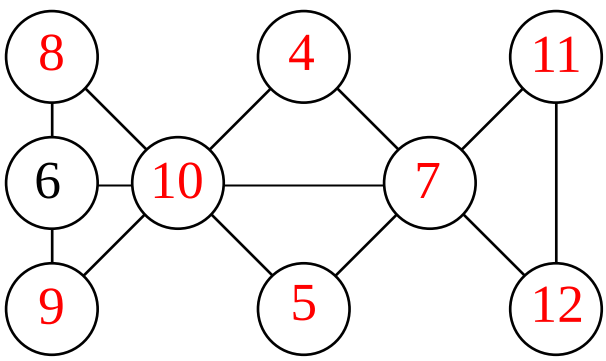


【Solution】

Each circle lies on two lines except for the two rimmed circles in the diagram below. The sum of the numbers inside them is 23－6=17. Hence the sum of the nine consecutive number is (7×23－17)÷2=72. The middle number is 72÷9=8, so that the nine numbers are 4, 5, 6, 7, 8, 9, 10, 11 and 12. The line on the right has only two circles. Hence the numbers inside must be 11 and 12, with 11 on top. The remaining numbers in these two lines add up to 12 and 11 respectively, and we must have 11=4+7 since 6 is already used. Now 12=5+7=4+8, so that the number in the rimmed circle on the right must contain 4 or 7. It cannot be 4 as the numbers in the two rimmed circles must add up to 17. Hence 7 is in the rimmed circle on the right and 10 is in the rimmed circle on the left. The placement of the remaining numbers are forced, completing the unique solution shown in the diagram below.

// totally correct [40pt]

// up-side-down or any other answer [0pt]



2. A clay tablet consists of a table of numbers, part of which is shown in the diagram below on the left. The first column consists of consecutive numbers starting from 0. In the first row, each subsequent number is obtained from the preceding one by adding 1. In the second row, each subsequent number is obtained from the preceding one by adding 2. In the third row, each subsequent number is obtained from the preceding one by adding 3, and so on. The tablet falls down and breaks up into pieces, which are swept away except for the two shown in the diagram below on the right in magnified forms, each with a smudged square. What is the sum of the two numbers on these two squares?

0	1	2	3	4	5	
1	3	5	7	9	11	
2	5	8	11	14	17	
3	7	11	15	19	23	
4	9	14	19	24	29	
5	11	17	23	29	35	

?	2012	2023

2012
2683
?

【Solution】

The smudged number in the first piece is 2012 − (2023 − 2012) = 2001. Note that the table is symmetric about the main diagonal. Hence the smudged number in the second piece is 2683 + (2683 − 2012) = 3354. The desired sum is therefore 2001+3354=5355

// "2001" for left blank [+5pt]

// "3354" for right blank [+10pt]

// correct answer [+25pt]

// a proof without correct answer [-5pt]

ANS: 5355

3. In a row of numbers, each is either 2012 or 1. The first number is 2012. There is exactly one 1 between the first 2012 and the second 2012. There are exactly two 1s between the second 2012 and the third 2012. There are exactly three 1s between the third 2012 and the fourth 2012, and so on. What is the sum of the first 2012 numbers in the row?

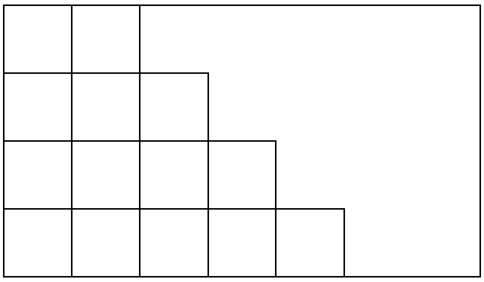
【Solution】

Let us break the row down into a staircase, with 2 numbers in the first, 3 numbers in the second, 4 numbers in the third, and so on, as shown in the diagram below on the left. Let us first calculate the total number of numbers in the first four rows. Make a second copy of the staircase, turn it upside down and put it together with the original copy to form a rectangle, as shown in the diagram below on the right. This rectangle has 3 more columns than rows. Hence it contains 4 × 7 = 28 numbers, which means that the original staircase contains 28 ÷ 2 = 14 numbers. The property that the rectangle contains 3 more columns than rows holds true regardless of the size of the staircase. We want the number of numbers in the rectangle to be as close to 2×2012 = 4024 as possible, without going over. Now 61×64 = 3904 while 62 × 65 = 4030. Hence we take a staircase with 61 rows, containing 3904 ÷ 2 = 1952 numbers. To bring the total up to 2012, we use an incomplete 62nd rows. Thus among the first 2012 numbers in the original row, there are 62 copies of 2012 and 2012−62 = 1950 copies of 1, for a total of 62 × 2012 + 1950 = 126694.

// totally correct [40pt]

// any other answer [0pt]

2012	1			
2012	1	1		
2012	1	1	1	
2012	1	1	1	1



ANS: 126694

4. In a test, one-third of the questions were answered incorrectly by Andrea and 7 questions were answered incorrectly by Barbara. One fifth of the questions were answered incorrectly by both of them. What was the maximum number of questions which were answered correctly by both of them?

【Solution】

The fraction of the questions answered incorrectly only by Andrea was $\frac{1}{3}-\frac{1}{5}=\frac{2}{15}$. Hence the number of questions must be a multiple of 15. Since answered 7 questions incorrectly, at most 7 questions may be answered incorrectly by both of them, so that the total number of questions is at most $7\div\frac{1}{5}=35$. To get the maximum number of questions answered correctly by both of them, we take the largest possible number of questions, which is 30. Then $30\times\frac{1}{5}=6$ questions were answered incorrectly by both, 7－6 = 1 question answered incorrectly by Barbara only, and $30\times\frac{2}{15}=4$ questions answered incorrectly by Andrea only. The number of questions answered correctly by both of them is therefore 30－6－1－4=19.

// (#all questions) is a multiple of 15 [+10pt]

// (#all questions) has upper bound 35 [+10pt]

// describe the results of the test [+10pt]

// correct answer [+5pt]

// check the case of “(#all questions)=15” [+5pt]

ANS: 19

5. Five different positive integers are multiplied two at a time, yielding ten products. The smallest product is 28, the largest product is 240 and 128 is also one of the products. What is the sum of these five numbers?

【Solution】

Note that 28 is the product of the smallest two numbers while 240 is the product of the largest two numbers. Hence the smallest two numbers are 1 and 28, 2 and 14 or 4 and 7. Note that the second smallest number is no less than 7, so that the second largest number is no less than 9. Hence the largest two numbers are 10 and 24, 12 and 20 or 15 and 16. Note that the second largest number is no greater than 15, so that the second smallest number is no greater than 13. Hence the smallest two numbers are 4 and 7. Now 128 is not divisible by 7, or by any of the possible values for the second largest number, namely 10, 12 and 15. The smallest number 4 cannot be one of its factors as the other factor 32 is greater than any possible value of the largest number. It follows that 128 is the product of the middle and the largest number. The largest number must be 16 as neither 20 or 24 divides 128. Hence the five numbers are 4, 7, 8, 15 and 16, and their sum is 50.

// totally correct [40pt]

// any other answer [0pt]

ANS: 50

6. The diagram below shows a square *MNPQ* inside a rectangle *ABCD* where $AB - BC = 7$ cm. The sides of the rectangle parallel to the sides of the square. If the total area of *ABNM* and *CDQP* is 123 cm^2 and the total area of *ADQM* and *BCPN* is 312 cm^2, what is the area of *MNPQ* in cm^2?

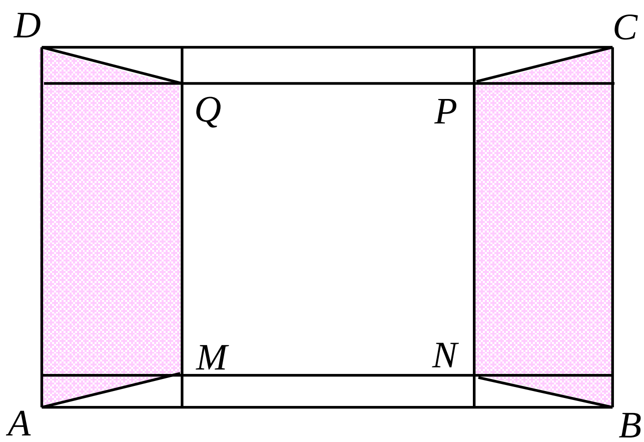
【Solution】

The diagram below is to be regarded as a frame with a square hole in the middle. The difference between the area of the shaded part of the frame and the area of the unshaded part of the frame is 312 − 123 = 189 cm^2. This difference is unchanged if we disregard the eight congruent right triangles at the four corners. The remaining parts consist of two shaded rectangles and two unshaded rectangles, whose heights are all equal to *MN*. Since the difference between their combined widths is 7 cm, we have *MN* = 189 ÷ 7 = 27 cm, so that the area of *MNPQ* is $27^2 = 729$ cm^2.

// move MNPQ to a corner of ABCD [+10pt]

// correct answer [+10pt]

// correct answer with proof [40pt]



ANS: 729 cm^2

7. Two companies have the same number of employees. The first company hires new employees so that its workforce is 11 times its original size. The second company lays off 11 employees. After the change, the number of employees in the first company is a multiple of the number of employees in the second

company. What is the maximum number of employees in each company before the change?

【Solution】

Suppose the number of employees in each company originally is not a multiple of 11. After the change, the number of employees in the second company divides the original number of employees, and must therefore divide the difference, which is 11. Since it cannot be a multiple of 11, it must be equal to 1, so that originally each company has 12 employees. Suppose the number of employees in each company originally is a multiple of 11. Divide each company into 11 branches of equal size. After the change, the number of employees in each branch of the second company divides 11 times the number of employees in each branch originally. Hence it must be a divisor of 11. If it is 1, then originally each branch has 2 employees and each company has 22 employees. If it is 11, then originally each branch has 12 employees and each company has 132 employees. Thus the maximum number is 132.

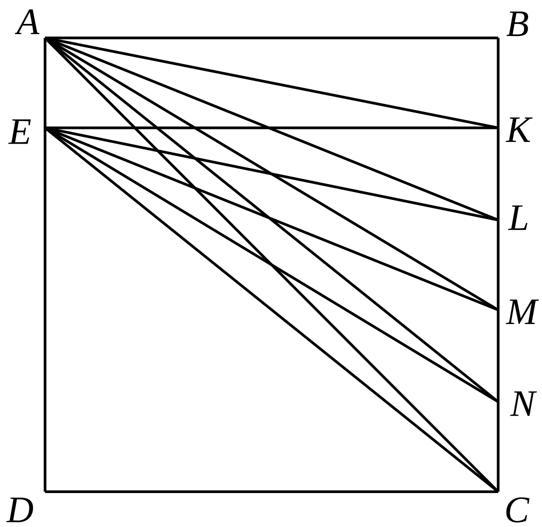
// totally correct [40pt]

// any other answer [0pt]

ANS: 132

8. *ABCD* is a square. *K*, *L*, *M* and *N* are points on *BC* such that *BK* = *KL* = *LM* = *MN* =*NC*. *E* is the point on *AD* such that *AE* = *BK*. In degrees, what is the measure of

$$\angle AKE + \angle ALE + \angle AME + \angle ANE + \angle ACE\ ?$$



【Solution】

Note that *EABK*, *EAKL*, *EALM*, *EAMN* and *EANC* are all parallelograms since each has a pair of equal and parallel opposite sides. It follows that $\angle AKE = \angle KAB$, $\angle ALE = \angle LAK$, $\angle AME = \angle MAL$, $\angle ANE = \angle NAM$ and $\angle ACE = \angle CAN$. Hence $\angle AKE + \angle ALE + \angle AME + \angle ANE + \angle ACE = \angle BAC = 45°$

// correct answer [+10pt]

// correct answer with proof [40pt]

ANS: 45°

9. The numbers 1 and 8 have been put into two squares of a 3×3 table, as shown in the diagram below. The remaining seven squares are to be filled with the numbers 2, 3, 4, 5, 6, 7 and 9, using each exactly once, such that the sum of the

numbers is the same in any of the four 2×2 subtables shaded in the diagram below. Find all possible solutions.

1		
		8

1		
		8

1		
		8

1		
		8

【Solution】

By considering the first two shaded subtables, we see that the number below 1 and the number above 8 must differ by 8 − 1 = 7. They can only be 9 and 2 respectively, as shown in the diagram below. By considering the last two shaded subtables, we see that the numbers below 9 and 8 must be consecutive. They can only be 3 and 4, 4 and 5, 5 and 6 or 6 and 7, as shown in the diagram below. It is easy to complete the table in the first three cases. In the last case, the difference between the top number and the bottom number in the middle column must be 6 − 1 = 5 = 7 − 2, but the maximum difference among the remaining numbers, namely 3, 4 and 5, is only 2. Thus there are only three solutions.

// totally correct [40pt]
// miss any solution [0pt]
// any other answer [0pt]

1	7	2
9	6	8
3	5	4

1	6	2
9	7	8
4	3	5

1	7	2
9	4	8
5	3	6

1		2
9		8
6		7

10. At the beginning of each month, an adult red ant gives birth to three baby black ants. An adult black ant eats one baby black ant, gives birth to three baby red ants, and then dies. During the month, baby ants become adult ants, and the cycle continues. If there are 9000000 red ants and 1000000 black ants on Christmas day, what was the difference between the number of red ants and the number of black ants on Christmas day a year ago?

【Solution】

Let us consider the cycle two months at a time. The number of new red ants in the first month is 3 times the original number of black ants. The number of new red ants in the second month is 3 times the number of surviving baby black ants in the first month. This number is 3 times the original number of red ants minus the original number of black ants. Hence in two months, the net gain in the number of red ants is 9 times the original number of red ants. In other words, the number of red ants is 10 times the number two months ago. The number black ants in the first month is 3 times the original number of red ants minus the original number of black ants. The number of baby black ants born at the beginning of the second month is 3 times the original number of red ants and 9 times the original number of black ants. Of these, 3 times the original number of red ants minus the original number of black ants are eaten. Hence the number of black ants is also 10 times the number two months ago. In a year, the number of red ants increases six times, to 1000000 times the original number. Since there are 9000000 red ants and 1000000 black ants on Christmas day, there were 9 red ants and 1 black ant on Christmas day a year ago, and the difference was 8.

// correct answer [+10pt]

// partial score from the following four cases must not exceed [+25pt]

// calculate the situation of “next month” correctly [+10pt]

// calculate the situation of “next two month” correctly [+10pt]

// calculate the situation of “last month” correctly [+10pt]

// calculate the situation of “last two month” correctly [+15pt]

// “10x” every two month [+5pt]

// correct answer with proof [40pt]

// a proof without correct answer [-5pt]

ANS: 8