"World Conference on the Mathematically Gifted Students - the role of Educators and Parents."

Content & Schedule

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Chairperson--- Dr. Elena R. Ruiz, Assistant Secretary for Programs and Projects, Republic of the Philippines

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Chairperson--- Prof. Yeong-Nan Yeh, Research Fellow of Institute of Mathematics, Academia Sinica, Taiwan

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Mathematics Lecture

*Tessellations with Regular Polygons*

Professor Andy Liu
Chairman, Academic Committee of IMC
Professor, Department of Mathematical and Statistical Sciences University of Alberta, Canada.
The good, the bad and the pleasure (not pressure!) of mathematics competitions

Lecturer: SIU Man Keung (蕭文強)
Department of Mathematics, University of Hong Kong, Hong Kong SAR, CHINA.

Introduction

A mathematics competition, as but one among many different kinds of extracurricular activity, should enhance the teaching and learning of mathematics in a positive way rather than present a controversy in a negative way. But why does it sometimes become a controversial issue for some people as to its negative effect? What are the pros and cons of this extracurricular activity known as a mathematics competition?

I cannot claim myself to be actively involved in mathematics competitions, but will attempt to share some of my views of this activity gleaned from the limited experience gained in the past years. The intention is to invite more discussion of the topic from those who are much more experienced and know much more about mathematics competitions. I thank the organizers for inviting me to give the talk so that I can take this opportunity to organize my thoughts and share them with you.

To begin with I like to state clearly which aspect of this activity I will not touch on in this talk. In the recent decade mathematics competitions have mushroomed into an industry, some of which are connected with profit-making or fame-gaining intention. Even if these activities may have indirect benefit to the learning of mathematics, which I seriously doubt, an academic discussion of the phenomenon is, mathematically speaking, irrelevant. Rather, it is more a topic of discussion for its social and cultural aspect, namely, what makes parents push their children to these competitions and to training centres which are set up to prepare the children for these competitions, sometimes even against the liking of the children? With such a disclaimer let me get back to issues on mathematics competitions that have to do with mathematics and mathematics education.

The “good” of mathematics competitions

The only experience I had of an international mathematics competition occurred in 1988 and in 1994. I helped as a coach when Hong Kong first entered the 29th IMO (International Mathematical Olympiad) in 1988 held in Canberra and I worked as a coordinator to grade the answer scripts of contestants in the 35th IMO in 1994 held in Hong Kong. Through working in these two instances I began to see how the IMO can exert good influence on the educational side, which I overlooked before. I wrote up my reflections on the IMO in an article [M.K. Siu, Some reflections of a coordinator on the IMO, Mathematics Competitions, 8(1) (1995), pp. 73-77.], from which I now extract the three points on “the good” of mathematics competitions.
“(1) All contestants know that clear and logical presentation is a necessary condition for a high score. When I read their answers I had a markedly different feeling from that I have in reading the answer scripts of many of my students. I felt cozy. Even for incomplete or incorrect answers I could still see where the writing is leading me. But in the case of many of my students, despite advice, coaxing, plea, protest (anything short of threat!) on my part, they write down anything that comes to their minds, disconnected, disorganized and perhaps irrelevant pieces. One possible reason for this bad habit is the examination strategy they have adopted since their school days --- write down everything you can remember, for you will score certain marks for certain key points (even if these key points are not necessarily presented in a correct logical order!) and the kind-hearted examiner will take the trouble to sift the wheat from the chaff! Many undergraduates still follow this strategy. Correct or incorrect answer aside, the least we can ask of our students should be clear communication in mathematics (but sadly we cannot).

(2) All contestants know that one can afford to spend up to one and half hour on each problem on the average, and hence nobody expects to solve a problem in a matter of minutes. As a result, most contestants possess the tenacity and the assiduity required of in problem solving. They will not give up easily, but will try all ways and means to probe the problem, to view it from different angles, and to explore through particular examples or experimental data. On the contrary, many of my students, again too much conditioned by examination techniques since their school days, would abandon a problem once they discover that it cannot be disposed of readily by routine means. In an examination when one races against time, this technique may have its excuse. Unfortunately, many students bring the same habit into their daily study. Any problem that cannot be disposed of in 3 minutes is a difficult problem and is beyond one’s capacity, hence no time should be wasted in thinking about it! This kind of ‘instant learning’ is detrimental to the acquirement of true understanding and it kills curiosity, thence along with it the pleasure of study.

(3) Some contestants have the commendable habit of writing down not only the mathematics, but remark on their progress as well. Some would write down that they could not go from that point on, or what they did so far seemed to lead nowhere, or that they decided to try a new approach. I really appreciate the manifestation of this kind of ‘academic sincerity’. (It is ironic to note that some leaders or deputy leaders tried to argue that those contestants were almost near to the solution and should therefore be credited with a higher score. They might be, but they did not, and the good intention of the leaders or deputy leaders is like filling in between the lines for the contestants.) On the contrary, some of my students behave in the opposite. They write down the given in the first line (amounting to copying the first part of the question), and write down the conclusion at the end (amounting to copying the final part of the question), then fill in between with disconnected pieces of information which may be relevant or irrelevant, ending with an unfounded assertion “hence we conclude …”! I am deeply
disappointed at this kind of insincerity, passing off gibberish as an answer. I would have felt less disappointed if the student does not know the answer at all."

The “bad” of mathematics competitions

Despite what has been said in the previous section I have one worry about mathematics competitions which has to do with the way of studying mathematics and doing mathematics, even more so for those who are doing well in mathematics competitions. Those who do well in mathematics competitions tend to develop a liking for solving problems by very clever but sometimes quite *ad hoc* means, but lack the patience to do things in a systematic but hard way or view things in a more global manner. They tend to look for problems that are already well-posed for them and they are not accustomed to dealing with vague situations. Pursuing mathematical research is not just to obtain a prescribed answer but to explore a situation in order to understand it as much as one can. It is far more important to be able to raise a good question than to be able to solve a problem set by somebody else who knows the answer already. One may even change the problem (by imposing more conditions or relaxing the demand) in order to make progress. This is, unfortunately, not what a contestant is allowed to do in a mathematics competition!

Of course, many strong contestants in various mathematics competitions go on to become outstanding mathematicians, but many stay at the level of being good competition problem solvers even if they go on to pursue mathematics. Many leave the field altogether. That is not a problem in itself, because everybody has his or her own aspiration and interest, and there is no need for everybody to become a research mathematician. On the other hand, it would be a pity if they leave the field because they get tired of the subject or acquire a lopsided view of the subject as a result of over-training during the youthful years they spent on mathematics competitions.

Looking at the history of several famed mathematics competitions we see a host of winners in the Eötvös Mathematics Competition of Hungary, started in 1894, went on to become eminent mathematicians; we see many medalists in the IMO’s, since the event started in 1959, received in their subsequent career various awards for their important contribution to the field of mathematics, including the Fields Medal, Navanlinna Prize, Wolf Prize,…; we see the same happens for many Putnam Fellows in the William Lowell Putnam Mathematical Competition in the USA for undergraduates. On the other hand, the Fields Medalist, Crafoord Prize and Wolf Prize recipient, Shing-Tung Yau, is noted for his public view against mathematics competitions. The eminent Russian mathematician of the last century, Pavel Segeevich Aleksandrov (1896-1982), was reported to have once said that he would not have become the mathematician he was had he joined the Mathematical Olympiad! An explanation of this polarity in opinions is to be sought in the way how one regards this activity known as a mathematics competition from the impression one gets in witnessing how it is run.
When I worked as a coordinator in the 35\textsuperscript{th} IMO in 1994 I noticed that some teams scored rather high marks, but all six contestants in the team worked out the problem in the same way, indicating solid training on the team’s part. However, there were some teams, not all of whose members scored as high marks, but each of whom approached the same problem in a different way, indicating a free and active mind that works independently and imaginatively. It made me wonder: will such qualities like independence and imagination be hampered by over-training, and if so, does that mean over-training for mathematics competitions defeats the purpose of this otherwise meaningful activity? Rather than over-training would an extended follow-up investigation of a competition problem enable the youngsters to better appreciate what mathematical exploration is about? I am sure many contestants who go on to become outstanding mathematicians followed this practice of follow-up investigation during the youthful years they spent on mathematics competitions.

I will now illustrate with two examples. The first example is a rather well-known problem in one IMO. We will see how one can view it as more than just a competition problem begging for just an answer. The other example is on a research topic where the main problem is still open to this date (as far as I am aware of). We will see how a research problem differs from a problem viewed in the context of a mathematics competition problem.

It was natural that I paid some special attention to the questions set in the 29\textsuperscript{th} IMO, although I did not take part in the actual event in July of 1988. Question 6 of the 29\textsuperscript{th} International Mathematical Olympiad reads:

“Let $a$ and $b$ be positive integers such that $ab + 1$ divides $a^2 + b^2$. Show that $\frac{a^2 + b^2}{ab + 1}$ is the square of an integer.”

A slick solution to this problem, offered by a Bulgarian youngster (Emanouil Atanassov) who received a special prize for it, starts by supposing that $k = \frac{a^2 + b^2}{ab + 1}$ is not a perfect square and rewriting the expression in the form $a^2 - kab + b^2 = k$, where $k$ is a given positive integer (*) .

Note that for any integral pair $(a, b)$ satisfying (*) we have $ab \geq 0$ or else $ab \leq -1$, and $a^2 + b^2 = k(ab + 1) \leq 0$, implying that $a=b=0$ so that $k=0!$ Furthermore, since $k$ is not a perfect square, we have $ab > 0$, that is, none of $a$ or $b$ is 0. Let $(a, b)$ be an integral pair satisfying (*) with $a > 0$ (and hence $b > 0$) and $a + b$ smallest. We may assume $a > b$. (By symmetry we may assume $a \geq b$ Note that $a-b$ or else $k$ is a number lying strictly between 1 and 2!) Regarding (*) as a quadratic equation with a positive root $a$ and another root $a'$, we see that $a + a' = kb$ and $aa' = b^2 - k$. Hence $a'$ is also an integer and $(a', b)$ is an integral pair satisfying (*). Since $ab > 0$ and
$b > 0$, we have $a' > 0$. But

$$a' = \frac{b^2 - k}{a} \leq \frac{b^2 - 1}{a} \leq \frac{a^2 - 1}{a} < a,$$

so that $a' + b < a + b$, contradicting the choice of $(a, b)$! This proves that $\frac{a^2 + b^2}{ab + 1}$ must be the square of an integer. [Having no access to the original answer script I try to reconstruct the proof based on the information provided by secondary sources. The underlying key ideas are (i) choice of a minimal solution, and (ii) the expression (*) viewed in the context of a quadratic equation.]

Slick as the proof is, it also invites a couple of queries. (1) What makes one suspect that $\frac{a^2 + b^2}{ab + 1}$ is the square of an integer? (2) The argument by *reductio ad absurdum* should hinge crucially upon the condition that $k$ is not a perfect square. In the proof this condition seems to have slipped in casually so that one does not see what really goes wrong if $k$ is *not* a perfect square. More pertinently, this proof *by contradiction* has not *explained* why $\frac{a^2 + b^2}{ab + 1}$ must be a perfect square, even though it *confirms* that it is so.

In contrast let us look at a less elegant solution, which is my own attempt. When I first heard of the problem, I was on a trip in Europe and had a ‘false insight’ by putting $a = N^3$ and $b = N$ so that

$$a^2 + b^2 = N^2(N^4 + 1) = N^2(ab + 1).$$

Under the impression that any integral solution $(a, b, k)$ of $k = \frac{a^2 + b^2}{ab + 1}$ is of the form $(N^3, N, N^2)$. I formulated a strategy of trying to deduce from $a^2 + b^2 = k(ab + 1)$ the equality

$$[a - (3b^2 - 3b + 1)]^2 + [b - 1]^2 = [k - (2b - 1)] \{ [a - (3b^2 - 3b + 1)](b - 1) + 1 \}.$$

Were I able to achieve that, then I could have reduced $b$ in steps of one until I got down to the equation $k = \frac{a^2 + 1}{a + 1}$ for which $a = k = 1$. By reversing steps I would have solved the problem. I tried to carry out this strategy while I was travelling on a train, but to no avail. Upon returning home I could resort to systematic brute-force checking and look for some actual solutions, resulting in a (partial) list shown below.

\[
\begin{array}{cccccccccccc}
a & 1 & 8 & 27 & 30 & 64 & 112 & 125 & 216 & 240 & 343 & 418 & 512 & \ldots \\
b & 1 & 2 & 3 & 8 & 4 & 30 & 5 & 6 & 27 & 7 & 112 & 8 & \ldots \\
k & 1 & 4 & 9 & 4 & 16 & 4 & 25 & 36 & 9 & 49 & 4 & 64 & \ldots \\
\end{array}
\]
Then I saw that my ill-fated strategy was doomed to failure, because there are solutions other than those of the form \((N^3, N, N^2)\). However, not all was lost. When I stared at the pattern, I noticed that for a fixed \(k\), the solutions could be obtained recursively as \((a_i, b_i, k_i)\) with

\[
a_{i+1} = a_i k_i - b_i, \quad b_{i+1} = a_i, \quad k_{i+1} = k_i = k.
\]

It remained to carry out the verification. Once that was done, all became clear. There is a set of ‘basic solutions’ of the form \((N^3, N, N^2)\) where \(N \in \{1, 2, 3, \ldots \}\). All other solutions are obtained from a ‘basic solution’ recursively as described above. In particular, \(k = \frac{a^2 + b^2}{ab + 1}\) is the square of an integer. I feel that I understand the phenomenon much more than if I just learn from reading the slick proof.

[Thanks to Peter Shiu we can turn the indirect proof into a more transparent direct proof based on the same key ideas. Proceed as before and set \(c = \min(a, b)\) and \(d = \max(a, b)\). Look at the quadratic equation

\[
x^2 - kcx + (c^2 - k) = 0,
\]

for which \(d\) is a positive root with another root \(d'\). Since \(d + d' = kc\) and \(dd' = c^2 - k < c^2 \leq dc\), we know that \(d'\) is an integer less than \(c\) so that \(d'c < 2c \leq a + b\). By the choice of \((a, b)\), \(d'\) cannot be positive. On the other hand,

\[
(d + 1)(d' + 1) = dd' + (d + d') + 1 = (c^2 - k) + kc + 1 = c^2 + (c - 1)k + 1 \geq c^2 + 1 > 0.
\]

Therefore, \(d' + 1 > 0\), implying that \(d' = 0\). Hence, \(k = c^2 - dd' = c^2\) is the square of an integer.]

The next example is a research problem on the so-called Barker sequence, which is a binary sequence of length \(s\) for which the sequence and each off-phase shift of itself differ by at most one place of coinciding entries and non-coinciding entries in their overlapping part. Technically speaking we say that the sequence has its aperiodic autocorrelation function having absolute value 0 or 1 at all off-phase values. For instance, 1101 is a Barker sequence of length 5, while 11010 of length 6 is not a Barker sequence. Neither is 11101011 of length 8 a Barker sequence, but 1110010010 is a Barker sequence of length 11. For application in group synchronization digital system in communication science R.H. Barker first introduced the notion in 1953. Such sequences for \(s\) equal to 2, 3, 4, 5, 7, 11, 13 were soon discovered and in 1961 R. Turyn and J. Storer proved that there is no Barker sequence of odd length \(s\) larger than 13. A well-known conjecture in combinatorial designs says that there is no Barker sequence of length \(s\) larger than 13, which has withstood the effort of many able mathematicians for more than half a century. Although the conjecture itself remains open, it has stimulated much research in combinatorial designs and in the design of sequences and arrays in communication science. In order to better understand the original problem researchers
change the problem and look at the 2-dimensional analogue of arrays or even analogues in higher dimensions, or other variations such as non-binary sequences and arrays over an alphabet set with more than two elements, or instead of a single sequence a pair of sequences (known as Golay complementary sequence pair) satisfying some suitably formulated modification on their aperiodic autocorrelation functions. In this sense the problem, instead of looking like an interesting piece of curio, opens up new fields and generates new methods and techniques which prove useful elsewhere. (I would recommend an excellent survey paper to those who wish to know more about this topic: Jonathan Jedwab, What can be used instead of a Barker sequence? Contemporary Mathematics, 461 (2008), pp. 153-178.)

**School mathematics and “Olympiad mathematics”**

Since many mathematics competitions aim at testing the contestants’ ability in problem solving rather than their acquaintance with specific subject content knowledge, the problems are set in some general areas which can be made comprehensible to youngsters of that age group, independent of different school syllabi in different countries and regions. That would cover topics in elementary number theory, algebra, combinatorics, sequences, inequalities, functional equations, plane and solid geometry and the like. Gradually the term “Olympiad mathematics” is coined to refer to this conglomeration of topics. One question that I usually ponder over is this: why can’t this type of so-called “Olympiad mathematics” be made good use of in the classroom of school mathematics as well? If one aim of mathematics education is to let students know what the subject is about and to arouse their interest in it, then interesting non-routine problems should be able to play their part well when used to supplement the day-to-day teaching and learning. In the preface to Alice in Numberland: A Students’ Guide to the Enjoyment of Higher Mathematics (1988) the authors, John Baylis and Rod Haggarty remark, “The professional mathematician will be familiar with the idea that entertainment and serious intent are not incompatible: the problem for us is to ensure that our readers will enjoy the entertainment but not miss the mathematical point, […]”

By making use of "Olympiad mathematics" in the classroom I do not mean transplanting the competition problems directly there. Rather, I mean making use of the kind of topics, the spirit and the way the question is designed and formulated, even if the confine is to be within the official syllabus. The so-called "higher-order thinking" is (and should be) one of the objectives in school mathematics as well. Sometimes we may have underestimated the capability and the interest of our students in the classroom. It is not true that they only like routine (and hence usually regarded as “easy”?) material. Perhaps they lack the motivation to learn because they find the diluted content dull and are tired of it. Besides, good questions not just benefit the learners in the classroom; it is also a challenging task for the teachers to design good questions, thereby upgrading themselves in the process.

There is a well-known anecdote about the famous mathematician John von
Neumann (1903-1957). A friend of von Neumann once gave him a problem to solve. Two cyclists A and B at a distance 20 miles apart were approaching each other, each going at a speed of 10 miles per hour. A bee flew back and forth between A and B at a speed of 15 miles per hour, starting with A and back to A after meeting B, then back to B after meeting A, and so on. By the time the two cyclists met, how far had the bee travelled? In a flash von Neumann gave the answer --- 15 miles. His friend responded by saying that von Neumann must have already known the trick so that he gave the answer so fast. His friend had in mind the slick solution to this quickie, namely, that the cyclists met after one hour so that within that one hour the bee had travelled 15 miles. To his friend’s astonishment von Neumann said that he knew no trick but simply summed an infinite series! (I leave it as an exercise for you to find the answer by summing an infinite series.)

For me this anecdote is very instructive. For one thing, it tells me that different people may have different ways to go about solving a mathematical problem. There is no point in forcing everybody to solve it in just the same way you solve it. Likewise, there is no point in forcing everybody to learn mathematics in just the same way you learn it. This point dawned on me quite late in my teaching career. For a long time I thought a geometric explanation would make my class understand linear algebra in the easiest way, so I emphasized the geometric viewpoint along with an analytic explanation. I still continue to do that in class to this date, but it did occur to me one day that some students may prefer an analytic explanation because they have difficulty with geometric visualization. To von Neumann, who could carry out mental calculation with lightning speed, maybe an infinite series was the first thing that came up in his mind rather than the time spent by the cyclists meeting each other!

Secondly, both methods of solution have their separate merits. The method of first calculating when the cyclists met is slick and captures the key point of the problem, killing it in one quick and direct shot. The other method of summing an infinite series, which is slower (but not for von Neumann!) and is seemingly more cumbersome and not as clever, goes about solving the problem in a systematic manner, resorting even to brute force. It indicates patience, determination, down-to-earth approach and meticulous care. Besides, it can help to consolidate some basic skills and nurture in a student a good working habit.

It makes me think that there are two approaches in doing mathematics. To give a military analogue one is like positional warfare and the other guerrilla warfare. The first approach, which has been going on in the classrooms of most schools and universities, is to present the subject in a systematically organized and carefully designed format supplemented with exercises and problems. The other approach, which goes on more predominantly in the training for mathematics competitions, is to confront students with various kinds of problems and train them to look for points of attack, thereby accumulating a host of tricks and strategies. Each approach has its separate merit and they supplement and complement each other. Just as in positional warfare flexibility and
spontaneity are called for, while in guerrilla warfare careful prior preparation and groundwork are needed, in the teaching and learning of mathematics we should not just teach tricks and strategies to solve special type of problems or just spend time on explaining the general theory and working on problems that are amenable to routine means. We should let the two approaches supplement and complement each other in our classrooms. In the biography of the famous Chinese general and national hero of the Southern Song Dynasty, Yue Fei (岳飛 1103-1142) we find the description: “陣而後戰，兵法之常。運用之妙，存乎一心。 (Setting up the battle formation is the routine of art of war. Manoeuvring the battle formation skillfully rest solely with the mind.)”

Sometimes the first approach may look quite plain and dull, compared with the excitement acquired from solving competition problems by the second approach. However, we should not overlook the significance of this seemingly bland approach, which can cover more general situations and turns out to be much more powerful than an *ad hoc* method which, slick as it is, solves only a special case. Of course, it is true that frequently a clever *ad hoc* method can develop into a powerful general method or can become a part of a larger picture. A classic case in point is the development of calculus in history. In ancient time, only masters in mathematics could calculate the area and volume of certain geometric figures, to name just a couple of them, Archimedes (c. 287 B.C.E. – c. 212 B.C. E.) and Liu Hui (劉徽 3rd century). Today we admire their ingenuity when we look at their clever solutions, but at the same time feel that it is rather beyond the capacity of an average student to do so. With the development of calculus since the seventeenth century and the eighteenth century, today even an average school pupil who has learnt the subject will have no problem in calculating the area of many geometric figures.

Let me further illustrate with one example, which is a competition problem posed to

\[ \begin{align*}
AB &= AC \\
\angle BAC &= 20^\circ \\
AD &= BC \\
\theta &= ?
\end{align*} \]

![Figure 1](image-url)
me by the father of a contestant. In isosceles, where \( AB = AC \) and the measure of \( \angle BAC \) being \( 20^\circ \), \( D \) is taken on the side \( AC \) such that \( AD=BC \). Find \( \theta \), the measure of \( \angle ADB \) (see Figure 1).

Clearly, if one is to employ the law of sines, then the answer can be readily obtained in a routine manner, namely,

\[
\frac{AD}{\sin(\alpha + \theta)} = \frac{AB}{\sin \theta}, \quad AD = BC = 2AB \sin \frac{\alpha}{2},
\]

where \( \alpha \) is the measure of \( \angle BAC \), thereby arriving at

\[
\tan \theta = \frac{\sin \alpha}{2\sin \frac{\alpha}{2} - \cos \alpha}.
\]

When \( \alpha = 20^\circ, \theta = 150^\circ \). However, the problem appeared in a primary school mathematics competition in which the contestant was not expected to possess the knowledge of the law of sines! Is there a way to avoid the use of this heavy machinery (for a primary school pupil)? I hit upon a solution by constructing an equilateral \( \triangle FBC \) with \( F \) inside the given \( \triangle ABC \). Pick a point \( E \) on \( AB \) such that \( AE = CD \) (see Figure 2).

Then it is not hard (by constructing \( DE, DF \)) to find out that the measure of \( \angle DBE \) is \( 10^\circ \) (Exercise) so that \( \theta = 180^\circ - 20^\circ - 10^\circ = 150^\circ \). Why would I throw in the equilateral as if by magic? It is because I had come across a similar-looking problem before: In an isosceles, where \( AB = AC \) and the measure of \( \angle BAC \) being \( 20^\circ \), points \( D \) and \( E \) are taken on \( AC, AB \) respectively such that the measure of \( \angle DBC \) is \( 70^\circ \) and that of \( \angle ECB \) is \( 50^\circ \); find \( \varphi \), the measure of \( \angle BDE \) (see Figure 3).
By constructing an equilateral $\triangle FBC$ with $F$ inside the given we can arrive at the answer $\varphi = 10^\circ$ (Exercise). These two versions are indeed the description of the same situation, because it can be proved that $AD = BC$ in the second problem. Only knowledge of congruence triangles suffices. No knowledge of trigonometry is required. However, if the measure of $\angle DBC$ and that of $\angle ECB$ are not 70° and 50° respectively, then the geometric proof completely breaks down! But we can still compute the measure of $\angle BDE$ by employing the law of sines, which is within what an average pupil learns in school. It has to be admitted that the method is routine and not as elegant, but it covers the general case and can be handled by an average pupil who has acquired that piece of knowledge.

**The pleasure (not pressure!) of mathematics competitions**

Before working as a coordinator for the 35th IMO I harboured a distrust of the value of mathematics competitions. I still harbour this distrust to some extent, all the more when I witnessed during coordination of the IMO in 1994 how some leaders or deputy leaders over-reacted out of too much concern for winning high scores. Putting strong emphasis on winning/losing will inculcate in the youngsters an unhealthy attitude towards the whole activity. Attaching undue importance to the competition by organizers, teachers, parents, students, is one main source that may cause distortion of the good intention of mathematics competitions, not to mention the more “commercial” consequences that take advantage of this misplaced emphasis. Not only it fails to bring about the ideal outcome of fostering genuine interest and enthusiasm in the subject, it takes the fun and meaning out of a truly extracurricular activity as well. Instead of pleasure we are imposing pressure on the youngsters.

Furthermore, the unilateral strengthening of ability to attain high score on these so-called “Olympiad mathematics” problems may have adverse effect on the overall growth of a youngster, not just in terms of academic pursuit in other disciplines (or in mathematics itself!) but even in terms of personal development. In particular, I am disappointed at not finding how mathematics competitions breathe life into a general mathematics culture in the local scene. On the contrary, many people may be misled into believing that those difficult “Olympiad mathematics” problems present the high point in mathematics, and that mathematics is therefore too difficult to lie within reach of an average person.

**Concluding remark**

On the whole I have great admiration for the talent of those youngsters who take part in a mathematics competition. What little I accomplish in trying out those competition problems with all my might they accomplish at a stroke, and explain it in a clear and lucid manner. I also have great respect for the dedication and enthusiasm of those organizers who believe in the value of a healthy mathematics competition. They
are serving the mathematical community in many ways.

A young friend of mine and currently a member of the Hong Kong IMO team, Andy Loo, by recounting his own experience since primary school days with mathematics competitions, highlights the essential “good” of mathematics competitions as lying in arousing a passion in the youngster and piquing his or her interest in the subject. I believe Andy is right. For those who finally do not benefit from this experience for one reason or another, perhaps it is just an indication that they lack a genuine and sustained passion for the subject of mathematics itself.

My good friend, Tony Gardiner, who is known for his rich experience in mathematics competitions and had served as the leader of the British IMO team four times, after reading my article in 1995 commented that I have missed the most significant point of mathematics competitions. He is correct in pointing out that one should not blame the negative aspects on the mathematics competition itself. He went on to enlighten me on one point, namely, a mathematics competition should be seen as just the tip of a very large, more interesting, iceberg, for it should provide an incentive for each country to establish a pyramid of activities for masses of interested students. It would be to the benefit of all to think about what other activities besides mathematics competitions can be organized to go along with it. These may include the setting up of a mathematics club or publishing a magazine to let interested youngsters share their enthusiasm and their ideas, organizing a problem session, holding contests in doing projects at various levels and to various depth, writing book reports and essays, producing cartoons, videos, software, toys, games, puzzles, … . I wish more people will see mathematics competitions in this light, in which case the negative impression, which I might have conveyed in this talk, will no longer linger on!

The good, the bad and the pleasure of mathematics competitions
Are to which we should pay our attention.
Benefit from the good; avoid the bad;
And soak in the pleasure.
Then we will find for ourselves satisfaction!
Mathematics Education in the Philippines

Lecturer: Elena R. Ruiz
Assistant Secretary
Department of Education, Republic of the Philippines

Mathematics is a subject that is integral to everyday life. There is Mathematics in practically everything one does. Thus, it is important for every learner to master the basics of mathematics and to be engaged in activities that call for critical thinking and problem solving.

This is the twin-goal of mathematics in basic education in the Philippines. The K to 12 basic education curriculum in Mathematics contains organized and vigorous content, a well-defined set of high-skills and processes, desirable values and attitudes and appropriate tools.

The curriculum covers five (5) content areas, namely: number and number sense, measurement, geometry, patterns and algebra and probability and statistics.

Specifically, it aims to develop among the learners the following skills:

- Knowing and understanding
- Estimating
- Computing and solving
- Visualizing and modeling
- Representing and communicating
- Conjecturing
- Reasoning
- Proving and decision-making
- Applying and connecting

Integrated with the teaching of Mathematical concepts is the honing of values of accuracy, creativity, objectivity, perseverance and productivity.

The aforementioned discussion is summed up in a conceptual framework:
The following is the conceptual framework of mathematics education in the Philippines.

To be able to develop the aforementioned skills and hone the desired values and attitudes, a shift from the traditional teaching, which is teacher-centered, to a student-centered instruction is necessary.

Traditionally, a mathematics class in the Philippines started with a teacher explaining the lesson for the day, asking about the lesson, engaging the learners in little or no discussion at all about the concept or topic to test their understanding and making them solve problems which were application of the lessons discussed. In other words, the most common strategies were exposition, practice and consolidation and discussion.

In the typical traditional mathematics classroom, more than half of the period was spent in individual problem solving which was challenging to some, grueling to others and frustrating to many. The result was lack of mastery of the desired learning competencies and general dislike for Mathematics.
The need to make teaching learner-centered has been gradually met in the Philippine classroom. The 2002 Basic Education Curriculum in Mathematics, particularly in the secondary level advocates using a variety of teaching strategies such as practical work, discussions, problem-solving investigations besides exposition and practice and consolidation (DepEd, 2002).

In a study conducted by UPNISMED, the teaching strategies found most effective were “hands-on” experience that brings students to their fullest learning capacity because they depend on themselves, cooperative learning because they can share better knowledge when they work together than when they are alone and self-discovery because it enhances students learning capacity (Peñano-Ho, 2004).

The K-12 Curriculum will be delivered utilizing these learner-engaging activities. Furthermore, it will be enhanced by using technology. Appropriate tools such as manipulative objects, measuring devices, calculating and computers, smart phones, tablet PC’s and the internet will be utilized. A study on the impact of using technology, specifically graphing calculators revealed that this tool reduces student anxiety in mathematics (Acelejado, De La Salle University).

The improving quality of Mathematics Education in the country is a result of this shift from teacher-centered to student centered teaching-learning approach or strategy. Furthermore, the Department of Education has been strengthening its teacher-capacity building program. There used to be so many non-Mathematics major teachers teaching mathematics. Now, there are just a few of them because the Department has trained practically all of them. The Department engaged the expertise of the different well-known teacher training institutions to handle the training of non-majors. In addition to this is the granting of three steps increment in the basic salary of teachers teaching mathematics – an initiative of Congress which includes granting of the same privilege to teachers of science. The Department’s ICT in Education program which provides computer units and internet connectivity to schools, secondary schools particularly is enhancing further the student learning. In keeping with the 21st century demand, integration of ICT in all learning areas is required.

What is very striking in the country is the very active participation of the government and non-government organizations engaged in mathematics education. There is the University of the Philippines Institute of Science and Mathematics Development (UPNISMED) which anchors its programs on the philosophy that “learners learn most effectively from experiences that are engaging, meaningful, challenging and relevant and from teachers who facilitate construction of knowledge from these experiences. “The institute has continuously been the DepEd’s partner in teacher training in mathematics and science. There is the Mathematical Society of the Philippines (MSP), a main professional organization for mathematicians in the country which aims “to promote interest of mathematics and its application, knowledge in mathematics and enable mathematical research. It runs an annual convention focused on
mathematical researches and educational issues.

There is the Mathematics Teachers Association of the Philippines (MTAP) which has a program in every school all over the country. It provides training for teachers and students – fast and slow learners alike. It is in strong partnership with METROBANK in its Annual Metrobank Math Challenge which provides opportunity to a lot of young math wizards to showcase their mathematical prowess. MTAP is already an institution in Philippine Education.

There is the Mathematics Teachers’ Guild, Philippines (MTG, Phil) which provides world-class training for mathematically-gifted learners in both levels, elementary and secondary, which enable them to participate in different international competitions, and which plays a very significant role in the building of the Philippine education image in the international fora and/or competition.

Improving the quality of mathematical education in the country remains a big challenge. The Department is optimistic that with the national government’s thrust of improving proficiency in math and science as one of its priority agenda and with the assistance from different mathematics organizations, the Department of Education will be able to implement its K to 12 mathematics curriculum effectively and successfully.

The formula for success will be simple:

\[ QME = SNG + GC + WTT + POS \]

where:

- **QME** = Quality Mathematics Education
- **SNG** = Strong National Government Support
- **GC** = Good Curriculum
- **WTT** = Well-trained Teacher
- **POS** = Private Organization Support
Mathematical Gifted Students – 
The Role of Educators and Parents Nigerian View

Lecturer: Dr (Prince) Adedibu Aderemi Abass FSTAN, MMAN
International Science, Technology, Mathematics & Engineering Coordinator for Africa/National JETS Coordinator

The Role of Parents and Educators
Mathematically Gifted Students; Education and Parents Role

Mathematics as the language of Science is spoken all over the world in diverse forms. The forms include business, academics, domestic, entertainment, communication etc. It is a language that recurs on daily basis and influences day-to-day activities in every stratum of human enterprises. Mathematics as a school subject is central to all academic disciplines. The globalization of knowledge via Information and Communications Technology (ICT) and Computer has added great value to the indispensability of mathematics in any of human endeavours.

As interesting as the discipline is academically, it is often widely dreaded by School children. The seeming “hatred” most students develop towards learning mathematics is a factor of many intrigues; parental background, peer group influence, fear, stereotype belief school factor and lack of determination as well as its uncompromising nature.

However, nature is funny at times in that it endows individuals severally. Natural endowment traits in humans are in categories, depending on its exhibition by the individual. Some individuals are regarded as being gifted. Hence, a mathematically gifted student is he/she who exhibit above excellent trait in the knowledge of mathematics. He/she goes beyond his/her Colleagues in appreciating mathematical principle in problem solving. As it is applicable to other academic disciplines, the mathematically gifted child is fast in reasoning/thinking, articulate in gathering, meticulous in applying and proactive in learning the subject with little encouragement. They are usually confident, bold and can study independently given a conducive environment both at home and at school.

Identification of Mathematically Gifted Children

Identifying this category of Students by parents at home and teachers/educators at school calls for careful/close observation of the traits aforementioned. A gifted Child in a particular discipline will usually be eager and disposed towards learning the subject, feel happy and enthusiastic when involved in the learning process, show too much preference/attention to the subject among others, participate actively during learning/teaching of the subject, motivate others and resourceful toward getting task accomplished. A careful and observant teacher should be able to recognize these traits in
his/her students within a very short time of his/her interaction with them in the classroom. Once noticed, the onus now rests on the teacher to explore all available avenues to encourage the child and his/her performance in and out of the school. The teacher’s monitoring ability become very much relevant at this stage. As a role model, he/she swings into action by putting out those behaviors that would elicit mentorship in him/her during classroom teaching. This will serve as background confidence for the child as he feels recognized and carried along with his/her peers. The educator/teacher has supportive role to play while mentoring a gifted child. It is altruism that human needs are insatiable; yes the teacher in this case should identify some of the needs of the student that must be satisfied for him/her to be fulfilled. These needs may be psychological, physical and material. For instance a gifted child from poor parental background may be grounded and his/her potential killed/buried when the acquisition of learning materials such as textbooks becomes a problem. Coming to the aid of the child through provision of essential learning materials has the potential to boost his/her morale and build hope in the child.

The learning environment where educators operate should be learner friendly enough to enhance development of potentials in gifted children. A gifted child in a hostile learning environment stands the risk of being denied avenue to develop his/her potential since this cannot be done in a vacuum.

The educator/teacher ensures that teaching facilities/materials are up-to-date and sufficient to meet the academic need of the gifted child. An obsolete and inadequate teaching/learning facilities is a potential gift killer! It tends to kill innovation, interest and morale.

Mentoring confer on the educator/teacher the need for him/her to be ahead or above board in terms of content knowledge of the subject, and pedagogical methodology. Otherwise, the gifted child’s ambition may be derailed. The teacher’s/educator’s resourcefulness is pertinent in the business of bringing a gifted child up no matter the discipline. Therefore, ability to provide alternatives (from locally available materials) create new things/ideas exhaust all available means/options before giving-up is the responsibility of the teacher.

In this ICT dominated age, it is obvious that for the educator to be fully and effectively perform his role on gifted children, we should be ICT compliant and ensure accessibility to ICT infrastructure by the gifted children to enable them reach out to the wider world.

Appropriate reward for the gifted child should be determined and provided by the teacher during classroom teaching to serve as motivation for the gifted child.

Parental Roles

Charity an adage says begins at home. Therefore, for a mathematically gifted child to develop his/her full potential, good parental care is paramount. The child must be
adequately cave and catered for by the parents. Otherwise the child vision may not see the light of the day.

Basic needs by the child such as uniform, books, shoes and feeding must be provided by the parents to ensure a smooth take-off from home. Just like the educator; parents would also need to monitor gifted child and put him/her through the right path of success, provide conducive environment for the child at home to study after school, ensure that enough time is provided for the child to engage him/herself in independent studies at home, show love to the child and his/her aspiration by being involved in mathematics related ventures that would boost the child’s morale; reciprocate the child’s commitment by extending appropriate reward to the child to motivate him/her.

Co-operation between the parents/educators in the area of discipline in-and-out of school is an important role the parent must play. This will add value to the mentoring role by the educator/teacher at school.

Parenting a gifted child in whatever area of academic discipline is a task that requires close attention to the child both at home and at school. This is because, if there is laxity loop-hole in either place, the child’s gift would be endangered and could be aborted before delivery. Hence, collaboration between the home and the school is vital to the development/growth of mathematically gifted children. The parents and educators has a vital role to play in checking the excesses of peer groups both at home and at school. This will forestall the possibility of influencing the gifted child negatively thus aborting his/her gift.

Above all, the spiritual upbringing at home and the complementary nurturing role by educators at school should not be compromised among the roles that are tangential to the development of the gifted child. In essence, the home and the school should bring-up the gifted child in the fear of God so that their God given gift will not be misused after all.
1.0 Introduction

1.1 Special Education and Mathematically Gifted

Mathematically gifted students have needs that differ in nature from those of other students. They require some differentiated instruction, defined by Tomlinson (1995) as "consistently using a variety of instructional approaches to modify content, process, and/or products in response to learning readiness and interest of academically diverse students." According to Greenes (1981), mathematically gifted students differ from the general group of students studying math in the following abilities: spontaneous formation of problems, flexibility in handling data, mental agility of fluency of ideas, data organization ability, originality of interpretation, ability to transfer ideas, and ability to generalize.

1.2 What should be done to differentiate curriculum, instruction and assessment for the mathematically gifted in the regular classroom?

Historically there has been debate about the role of acceleration versus enrichment as the differentiation mode for mathematics. Most experts recommend a combination. The following are suggestions for differentiating for the mathematically gifted by using (1) assessment, (2) curriculum materials, (2) instructional techniques, and (4) grouping models. These opportunities should be made broadly available to any student with interest in taking advantage of them.

- Give pre-assessments so that students who already know the material do not have to repeat it but may be provided with instruction and activities that are meaningful. In the elementary grades, gifted learners still need to know their basic facts. If they do not, don't hold them back from other more complex tasks, but continue to work concurrently on the basics.
- Create assessments that allow for differences in understanding, creativity, and accomplishment; give students a chance to show what they have learned. Ask students to explain their reasoning both orally and in writing.
- Choose textbooks that provide more enriched opportunities. Unfortunately, curriculum in this country is mainly driven by textbooks, which are used about 80% of the time (Lockwood, 1992). Math textbooks often repeat topics from year to year in the grades prior to algebra. Since most textbooks are written for the general population, they are not always appropriate for the gifted.
- Use multiple resources. No single text will adequately meet the needs of these learners.
- Be flexible in your expectations about pacing for different students. While some may be mastering basic skills, others may work on more advanced problems.
- Use inquiry-based, discovery learning approaches that emphasize open-ended problems with multiple solutions or multiple paths to solutions. Allow students to design their own ways to find the answers to complex questions. Gifted students may discover more than you thought was possible.
- Use lots of higher-level questions in justification and discussion of problems. Ask "why" and "what if" questions.
- Provide units, activities, or problems that extend beyond the normal curriculum. Offer challenging mathematical recreations such as puzzles and games.
- Provide AP level courses in calculus, statistics, and computer science or encourage prepared students to take classes at local colleges if the supply of courses at the high school has been exhausted.
- Differentiate assignments. It is not appropriate to give more problems of the same type to gifted students. You might give students a choice of a regular assignment; a different, more challenging one; or a task that is tailored to interests.
- Expect high level products (e.g., writing, proofs, projects, solutions to challenging problems).
- Provide opportunities to participate in contests and give feedback to students on their solutions. After the contests, use some of the problems as the basis for classroom discussions.
- Provide access to male and female mentors who represent diverse linguistic and cultural groups. They may be within the school system, volunteers from the community, or experts who agree to respond to questions by e-mail. Bring speakers into the classroom to explain how math has opened doors in their professions and careers.
- Provide some activities that can be done independently or in groups based on student choice. Be aware that if gifted students always work independently, they are gaining no more than they could do at home. They also need appropriate instruction, interaction with other gifted students, and regular feedback from the teacher.
- Provide useful concrete experiences. Even though gifted learners may be capable of abstraction and may move from concrete to abstract more rapidly, they still benefit from the use of manipulatives and "hands-on" activities.

1.3 What is the responsibility of schools and teachers in developing giftedness in mathematics?

Classroom teachers and school districts share the responsibility of addressing the needs of gifted students.

Teachers need training and support in recognizing and addressing the needs of the mathematically gifted.

Teachers who teach mathematics to gifted learners need a strong background in mathematics content. If the school has only a few students with special needs and does not have such a teacher, a mentor from outside the school should be located to work with individuals.
• A coordinated curriculum plan needs to be in place so that the mathematical experiences for students are not duplicated or interrupted from one year to the next.
• The school should have an organized support system that includes resource books, technology, and human resources.
• Regular mathematics classrooms that offer sufficiently challenging and broad experiences for gifted students have the potential to enrich the learning community as a whole since other students will be interested in attempting, perhaps with help, some of the more challenging tasks. If math classes offer diversity in assignments, products, and monitor student needs, all students will be able to work at their own challenge level.

1.4 Issues

Now that we have considered some of the options for educating talented youth in regular classroom, let us turn to some of the issues that the regular classroom teacher might encounter:

1.4.1 Students have varying abilities.

Since students’ abilities vary, programs offered to them should be varied; the curriculum be matched to the abilities of students by adjusting the pace and the depth at which the material is presented. Skipping a grade in science might be the most appropriate option for one student, while doing enrichment activities and independent study projects might be the most appropriate for another.

1.4.2 Students might be gifted in math, but not in other subjects.

Many students are gifted in math, but do not have equal strengths in other academic areas. In some cases, these students are not in their school’s gifted program. This makes sense if the gifted program is tailored to students gifted in verbal areas, but it is important not to deny mathematically talented students opportunities because they are not labeled “gifted”. This discussion holds true for students gifted in other academic areas as well.

1.4.3 The gifted program might not meet all of the mathematically talented students’ need in mathematics.

The gifted programs in many schools are verbally oriented, and little time during the academic year is devoted to the study of mathematics. The mathematics that is studied might be covered in a random fashion (for example, challenge problems and enrichment sheets unrelated to each other). The gifted programs will meet mathematically talented students’ needs only if the students are permitted to move ahead in the mathematics curriculum at an appropriate pace and depth, not if they are given random enrichment activities.
1.4.4 "Acceleration versus enrichment" is a false dichotomy.

Good acceleration contains some enrichment, while good enrichment is accelerative. Proper pacing and the opportunity to study the subject in depth are both needed for the curriculum to be matched to student’s ability.

1.4.5 Students can be extremely talented in mathematics, but still make mistakes in computations.

Studies have demonstrated that mathematically-talented youth perform significantly better on conceptual tests than on computational tests. These students seem to show a good intuitive grasp of mathematics, but they lack the same level of skill in computations. They might make mistakes in computations because they have developed bad habits such as not writing down their thought processes while solving problems. Perhaps their computational skills have not caught up to their advanced conceptual understanding of mathematics, because they have not learned the appropriate terminology or algorithms. These students should be challenged by learning new concepts while polishing their computational skills. They should not be held back because of a relative weakness in computations.

2.0 Special Education and Mathematically Gifted in Malaysia

2.1 Ministry of Education, Malaysia (Services provided for children with special needs)

The Malaysia Ministry of Education’s Special Education Program (Program Pendidikan Khas Kementerian Pendidikan Malaysia) consists of:

- Special Schools (Sekolah Khas) for students with vision and hearing disabilities
- Special Education Integration Program (Program Pendidikan Khas Integrasi) is provided for children with learning, hearing and vision disabilities. The Program is carried out in normal primary and secondary school, as well as in technical/vocational secondary schools that use the withdrawal and partially inclusive approach to teach and learn.

The Special Education Integration Program is managed by the State Department of Education (Jabatan Pendidikan Negeri) while the Special Education Department (Jabatan Pendidikan Khas) is in charge of issues pertaining to policies and content.

Special Education Program Curriculum: The curriculum used in Special Schools and the Special Education Integration Program are the National Curriculum and the Alternative Curriculum. Special education students participate in extra-curricular activities with normal students.

Assessment of Special Education Students: All special education students sit for public examinations such as the Ujian Penilaian Sekolah Rendah (UPSR), Penilaian Menengah Rendah (PMR) and Sijil Pelajaran Malaysia (SPM) except for those who are following the Alternative Curriculum.
For students under the Alternative Curriculum, those taking the Standard Kemahiran Kebangsaan (SKK) will be certified with the Sijil Kemahiran Malaysia, while those taking Art and Design Courses will be awarded the Sijil Perakuan Sekolah and the Sijil Khas Vokasional.

Special Education Rehabilitation Program:
- The Program used is called the Program Pendidikan Pemulihan Khas Tahap Satu (PKTS). Students are assessed for reading writing and mathematics skills using the IPP3M (Instrumen Penentu Penguasaan 3M). IPP3M is divided into IPP3M 1 (for Year 1 students), IPP3M 2 (for Year 2 students) and IPP3M 3 (for Year 3 students).
- The approach to teaching and learning used is the temporary withdrawal system.

Conditions of entry for students into the Special Education School Program are:
- Aged no less than 5 years (for Preschool Program)
- Aged 6+ to 14+ years (for Primary School Program)
- Aged 13+ to 19+ years (for Secondary School Program)
- Certified by medical doctor
- Can manage themselves (self-care) without the assistance of others.

Duration of schooling:
- Length of primary schooling for children with special needs is 6 years
- Length of secondary schooling for children with special needs is 5 years
- This duration can be extended for a maximum of 2 years at any level, whether primary or secondary, depending on the needs of the student.

All Special Primary Schools are academic-based. Facilities provided at primary school level include hostel facilities and free meals. Students that are under the Special Education Integration Program can take either the national curriculum or the alternative curriculum.

Secondary education provides either the academic-based or vocational-based choices:
- Special School: Sekolah Menengah Pendidikan Khas Persekutuan Pulau Pinang provides the MPV vocational subject as an elective. Sekolah Menengah Pendidikan Khas Vokasional Shah Alam offers the Sijil Kemahiran Malaysia.
- Special Education Integration Program (Hearing Impairment): Sekolah Menengah Vokasional (EAT) Azizah, Johor Baru; Sekolah Menengah Teknik Langkawi, Kedah; Sekolah Menengah Teknik Batu Pahat, Johor; Sekolah Menengah Vokasional Bagan Serai, Perak; Sekolah Menengah Teknik Tanah Merah, Kelantan; Sekolah Menengah Vokasional Keningau, Sabah

Special Education Services Centre: Established in 1999 for the purpose of providing one-stop specialist services to parents and students with special needs. This centre functions to support agencies involved with special needs. Services provided include: Audiological services, sign language classes, therapy activities, parent counseling, toy library and resource material.

2.2 Registration process for the Special Education Programme
Step 1: Start
Parents and teachers suspect the child of having a learning disability.

Step 2: Confirmation
Confirming that the child has a disability. Certification of the child’s disability by medical professionals in a government hospital, health care centre or private clinic for the purpose of obtaining placement, and to receive early intervention from the Ministry of Health Malaysia (Kementerian Kesihatan Malaysia).

Step 3: Registration
Register your disabled child with the State Education Department (Jabatan Pendidikan Negeri) in order to receive suitable education.
*Also register your disabled child with the Social Welfare Department (Jabatan Kebajikan Masyarakat) for the purpose of receiving suitable services.

Step 4: Placement
Students will be given the opportunity for placement in either Special School or the Special Education Integration Programme. Placement will be based on the nearest school or Special Education Programme.

Step 5: Probation Period
Students who have been placed in the Malaysian Ministry of Education’s Special Education Programme will be given a 3 month probation period.

Step 6: Confirmation
After the probation period, students who are successful in following the Special Education Program will be confirmed in the program. Students that fail to follow the Ministry’s Special Education Programme will be referred back to the Social Welfare Department to receive suitable rehabilitation through Community Based Rehabilitation.

3.0 A Case Study In Malaysia

3.1 Adi Putra, the Malaysian math genius
Six year old Adi Putra Abdul Ghani who surprised the nation with his ability to solve complex mathematical problems, began his formal education at Sekolah Kebangsaan (Primary School) Matang Buluh, here on January 2005.

The boy has to follow the normal curriculum like thousands of other Standard One pupils, said Education Director. He said the Education Ministry had taken a special interest in the boy and instructed the school to submit his performance record to the ministry every fortnight to assess his true ability.

"We acknowledge his extraordinary ability in mathematics to the extent that he has reached the standard of a Sijil Pelajaran Malaysia (SPM) holder (17 and 18 year-olds) but we also want to know his ability in other subjects," he told reporters when visiting the school. The Education Director said Adi Putra's overall development would be monitored closely so that the right assistance could be given to nurture his rare ability and if Adi Putra is determined to be a child genius, the Education Department will find ways to fastback his education.

He said a counseling teacher would guide the lad to mould his interest not only in mathematics but other fields as well and also help him develop personality.

But when Adi Putra was 10 years old unlike his peers, he does not go to an elementary school in Malaysia. As a boy who has a supernormal memory capacity, math skills and intelligence quotient, normal curricula are unfit for him. The 10-year-old Malaysian boy, endowed with an astounding ability to memorize numbers and do quick calculations, demonstrated his talents at his home in Taman Berlian, Gombak, Selangor State. He memorized a 32-digital number after reading it, and repeated it precisely after a while. Moreover, the boy repeated the number in reverse order after talking about other issues and calculating other math problems for nearly one hour. Adi can memorize figures of up to 40 digits in around two minutes and give the result for a complex mathematical question within a minute. Adi said he could memorize a number with 70 digits. He also demonstrated calculating skills. Using his own formula, he gave the correct answer to a multiplication of a five-digit number times a seven-digit one within minutes. Born on Jan. 27, 1999, in Perak State, the boy is said to have more than 200 formulas to solve difficult multiplication calculations in one or two minutes. In 2008, he enrolled in grade two at a primary school but soon switched to grade five. Still, he found the curriculum too simple, so he decided to find a teacher elsewhere who could teach him something new. Currently, Adi studies at home, mainly using the Internet. The subjects include math, physics, chemistry and biology. He also surfs the Internet to study business and trading subjects, while following the developments of overseas markets. Adi is the chief
executive officer of his own company called Aura Adi Enterprise, which sells products branded with his name, including medicine. Perhaps the youngest CEO in the country, he manages the company with the help of his mother, Serihana Elias. Adi loves swimming and fishing. However, as a supernormal child, he misses some fun with his peers.

According to Adi he wishes to return to school, saying though he found the school curricula too easy, he missed the part of spending time with his classmates. He missed interacting with his peers a lot as now he did not have too many companions at home. While enrolling in a university may sound like a good option, the shy boy said he was afraid that it would not be easy mingling with students so much older than him. Besides, he felt a bit uneasy over the fame and attention showered on him by mass media coverage. Adi’s talents have made him a sort of superstar in the country. Even former Malaysian Prime Minister Mahathir bin Mohamad and Abdullah Ahmad Badawi had met him. Adi’s father, Abdul Ghani Abdul Wahid, said his son somehow refused to watch television because he did not want to see himself on the screen. Adi also admitted that he felt somewhat under stress remaining in the limelight of the public. Because of his extraordinary abilities, he was invited to be a mentor at universities. He could earn RM 6,000 (1,714 U.S. dollars) an hour from such jobs, and he donated half of his income from the lectures to orphanages, his father said. In the future, Adi said his ambition is to become a professor.

3.2 Adi Putra’s Dilemma

Adi Putra the Math’s genius, 7 year old finds the public school system is boring and the government is unable to provide holistic education for not only someone of Adi’s standards, but also to other students as well.

Of course, bearing in mind that Adi’s present school is obviously at a lost of how to manage a child who is far advanced compared to his peers, the school had no choice but to continue teaching the standard curriculum for Primary One students. The teachers, unfortunately, cannot give Adi special treatment as that ill not sit well with his other classmates and by giving him preferential treatment, it will also make him an outcast among his peers who would become jealous of the special treatment.

The simple solution to this is simply for Adi to not go to school, after all, what's there for him to learn in public school when he can already read a newspaper at the age of three and solve Add Math’s problems by age 6.

So Adi feels bored and decided to cut classes. The Primary One curriculum is basically something like kindergarten all over again, where students are drilled on writing, reading and calculation, all of which Adi has already mastered. There is nothing
new to offer Adi in the curriculum or lessons, so why stay in school. The parents of Adi did the right thing by complying with his wishes to not go to school.

Still, what the school did, which is to threaten to expel him, could have been done with a bit more tact. It is obvious that the school's action in issuing show cause letters to the boy and his parents, warning him that if he does not show up in school, he will be expelled, was done with little tolerance and understanding of Adi's predicament. Normally expulsion are done to students with serious discipline problems and Adi is certainly not a boy with discipline problems. But the label would stick and he will have to live with the memory that he has been threatened with expulsion.

Did the school ever gave a thought to how Adi might feel and how his peers in school will think of him? Obviously not. All the school cares for is its policy to ensure that students attend classes. Anyone who does not is a problematic child. Thus, will be expelled. The school never thinks about the child's feelings. To them, that's secondary when it comes to doing their duty to uphold school rules and policies.

What the government should do, and if they are really serious and concerned for Adi's welfare and education, is to let the boy enroll in the Islamic International School here in KL, or any other private school that is better suited and prepared to manage a genius child like Adi. The most important thing here is not only to nurture his abilities so that he will not be one day burnt out like many other geniuses around the world, but also to build him up in other skills, not just Math's. He has expressed interest to learn other languages, and should be encouraged to do so.

Nurtured and taught the right way, we can expect great things from this boy in the future. But until then, it is also important that the glare of the media and attention be diverted away from him. He's still first and foremost a child and should not be exposed to such pressures. He should be allowed to enjoy the things every normal child enjoys. Every child have their right to have a normal childhood, to experience everything a normal child should experience, regardless of whether he's a genius or not.

True, expectations on him will be high in the future. but with proper care in his upbringing, Adi will become one of Malaysia's outstanding personality in the future.

According to the one of the Malaysian local newspaper, The Permata Negara programme sets the groundwork for a special education for gifted children endowed with high intelligence quotient (IQ), said Prime Minister Datuk Seri Najib Tun Razak. He said if the potentials of gifted children were not optimised in the best interest of the country, they would be denied of an education system that could enrich their high-order thinking that supersedes the normal level of intelligence. If a suitable education system is not provided for them, he said the children would get bored with lessons easily or they might be sent to study or work abroad by their parents. Najib said through the Permata Pintar programme, the children's ability could be moulded and enhanced to enable them to achieve extraordinary success.

3.3 Only 10 and genius Adi is a CEO and a Lecturer
CHILD genius Adi Putra Abdul Ghani, 10, is now the chief executive officer of two companies and a lecturer who charges RM6,000 per hour. His mother Serihana Alias operates the two companies, which sell vitamins under the brand Adi. Adi Putra, who is supposed to be attending Year Four classes at his age, has stopped schooling. He has been invited to certain local universities to give lectures and he wants to be a lecturer in Islamic studies. According to his mother Serihana he keeps track of foreign stock markets via the Internet and studies at home. He’s interested in mathematics, physics, chemistry, geography and biology, but not so much in history and politics. He dislikes reading books but loves spending his time browsing the Net for study materials.

3.4 At The Age of 12 Launched His First Mathematic Book

Adi Putra Abdul Ghani admits that it is a burden being called a "child genius".
"There are so many expectations, but it is a burden I have to carry," the child prodigy said at the 2011 Kuala Lumpur International Book Fair yesterday. Adi Putra, 12, surged into the limelight in 2009 when it was reported that he gave lectures at universities when he was supposed to be in Year Four. He has reportedly invented 234 new mathematical formulae and learnt eight languages. He was also accorded several awards, including the "This Decade's Muslim Mathematician" from Riau University of Indonesia in 2009. During the fair, Adi Putra launched his first book titled "Seni Matematik Islam" (Islamic Art of Mathematics) and wowed the audience by memorising 200 numbers within a minute. Adi Putra said the book was easy to grasp and would help youngsters gain a better understanding of mathematics and teach them calculations. "It can be utilised by children as young as six," he added. Adi Putra said he used his own insight into the world around him to come up with mathematical formulae like the hill' formula for multiplication. His mentor Assoc Prof Dr Pakharuddin Samin, from the mechanical engineering faculty of Universiti Teknologi Malaysia, said the book contained 12 chapters, with each chapter having a formula on mathematics Adi Putra came up with by himself. "These are basic formulae derived from his own creation and from great Muslim thinkers and mathematicians like Al-Khawarizmi," he added. Asked how other children can improve their mathematical prowess like himself, Adi Putra advised schoolgoers not to use calculators until they had memorised mathematical tables. "Calculators tend to make our brains lazy when used for purposes other than checking the results of an equation. "They should also play less video games. It may help them be creative but it is a distraction from actual study," he added.

References

Education for Mathematically Gifted Students in Taiwan

Lecturer: Prof. Cheng-Der Fuh
National Central University, Taiwan

Gifted students in mathematics require special education. In Taiwan, outside of the gifted classes in high schools all around the country (see Appendix A), three major education systems for these talented children are: the elite programs for talented students; the science fairs for students; and various mathematical competitions. These systems play a major part in Taiwan’s education for gifted students, which I summarize their histories and current status as follows.

1. Elite Programs

The first system is the elite programs. These programs, usually hosted by the government or universities, select outstanding students nationally or regionally, and provide them with additional courses and/or resources on top of their regular high school education.

One of the largest programs among them is the one jointly held by the Ministry of Education and the National Science Council, which has five sub-programs for talented students in Physics, Mathematics, Chemistry, Biology and Earth Science. In each sub-program, ten domestic scholars are invited to be the advisor of these students. Many other universities and foundations held similar programs. The following are just a few that have programs in Mathematics:

- Foundation of United Microelectronic Corporation
- National University of Kaohsiung
- Department of Applied Mathematics, National Sun Yat-sen University
- Department of Applied Mathematics, National Hsinchu University of Education

Students in these programs usually further participate in science fairs and/or mathematical competitions to further prove their talents in Mathematics, which we would introduce in the following.

2. Science Fairs

The second system is the science fairs for students, which are exhibition contests for student research projects. There are two major exhibitions in Taiwan. The first one is the annual “National Primary and High School Science Fair” hosted by the National Taiwan Science Education Center. Founded in 1960, this national contest opens to students from elementary school to high school, and with talent in different subjects as listed in Table 1:
<table>
<thead>
<tr>
<th>Age</th>
<th>Elementary School</th>
<th>Junior High School</th>
<th>Senior High School</th>
<th>High School of Commerce</th>
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<tr>
<td>Subjects</td>
<td>Physics</td>
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<td>Biology</td>
<td>Mathematics</td>
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<td>Biology</td>
<td>Mathematics</td>
<td>Earth Science</td>
</tr>
<tr>
<td></td>
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<td>Earth Science</td>
<td>Applied Science</td>
<td>Applied Science</td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td>High School of Commerce</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Mechanism</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>Electronics and Information Science</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
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<tr>
<td></td>
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<td>Agriculture and Biotechnology</td>
</tr>
</tbody>
</table>

*Table 1. Groups and Subjects for National Primary and High School Science Fair*

The scale of this science fair is so large that over 200,000 students and teachers participate the contests annually, with over 20,000 projects are presented each year, leaving a widespread and significant impact.

Many of these projects are so outstanding that even draws international attention. Therefore, in 1982, the International Science and Engineering Affair (ISEF) of the United State of America, one of the top science affairs in the world, invites Taiwanese student to the fair. At the beginning, two students are selected from the National Primary and High School Science Fair to participate ISEF. In 1989, due to the outstanding performance of Taiwanese students, the ISEF raises the number of students to six, selecting two students each from each of the northern, central and southern Taiwan. From that on, more and more international science fairs invite Taiwanese student to participate, with the number of invitations increasing rapidly that eventually exceeds the scope of the original National Primary and High School Science Fair.

Therefore, in 1991, another selecting fair was established to meet this need, which was later named as “Taiwan International Science Fair” in 2002. Each year, top students from grades nine through twelve gather at this fair to compete not only for the prize, but also the chance to represent Taiwan at various fairs around the world. The following are just some of these science fairs that have official invite Taiwanese students:

- Intel International Science and Engineering Fair (Intel ISEF)
- Expo Sciences International (ESI) in Slovakia
- Canada-Wide Science Fair (CWSF)
- Hong Kong Joint School Science Exhibition (HKJSSE)
- Singapore Science and Engineering Fair

The Taiwanese contestant performs extremely well in these international science affairs. For example, between 1982 and 2010, a total of 160 projects of Taiwanese students have been selected to compete at ISEF, in which they had received 1 Intel Young Scientists award, 9 Best of Category Awards, 92 Grand Awards and 106 Special Awards. The ratio of Taiwanese students to win a Grand Award is about 64% (102/160), which outperforms the ratio of all participants, which is around 30%.
Recently, two other science affairs arise that exist simultaneously with the existing two major affairs. They are:

- The Macronix Science Award (held by Macronix Foundation of Education)
- Shing-Tung Yau High School Mathematics Award (held by Department of Mathematics, National Taiwan University, and specialized in mathematics.)

An important feature of these two science affairs is that they are non-government held.

3. Mathematical Competitions

The Least but not last important system is the various mathematical competitions in Taiwan. There are many mathematics competitions simultaneously exist in Taiwan, which are usually held by government or educational foundations. Among these competitions, one that is generally considered by most students as the ultimate challenges is the national IMO selection competition. As its name indicates, this competition is designed to select the six students that represent Taiwan in the International Mathematics Olympiad (IMO), one of the top mathematical competitions in the world.

Taiwan began to participate in Mathematical Olympiad in the 1990’s. In November 1990, Prof. Shiing-Shen Chern suggested that, in order to train the mathematics strength of high school students, Taiwan should participate in International mathematical competitions, such as IMO. In December, the authority approved the suggestion, assigning Prof. Fon-Che Liu to contact the IMO, and also Prof. Jau-D Chen to contact the APMO (Asian Pacific Mathematics Olympiad). As a result, Taiwan begins to participate in APMO since 1991, and in IMO since 1992. Prof. Chen is in charge of Taiwan IMO program from 1992 to 2002. Prof. Cheng-Der Fuh is in charge of Taiwan IMO program from 2003 to present.

The current selection procedure of the contestants is as follows:

1. Candidates must first participate the APMO contest in March. In order to participate the contest, student must fulfill at least one of the following conditions:
   - Top 30 in the APMO preliminary contest held in January.
   - Outstanding students in the APMO camp held in February.
   - Previous contestant of APMO.

Based on the result of APMO, about 35 students are selected to join the IMO selection camp.

2. The IMO selection camp consists of three stages. After the first stage, about 15 students are selected to continue. After the second stage, about 10 students are selected to proceed. And the final six contestants for this year’s IMO will be determined after the third stage. Each stage last for five days with one week breaks between each of them.

3. After the selection camp is the training camp, which usually contains five stages. In
these stages, contestants are trained for the IMO competition in July.

So far, the selection and training procedure has function very well, leading Taiwan to be a strong competitor in IMO. In 2011, Team Taiwan achieved 2 gold medals and 4 silver medals, ends up in top 8. Among the higher records held by Team Taiwan are in 1998 and 2004, where we end up in 5th and 6th, respectively. For the results of other years, please see Appendix B.

Many other mathematical competitions stand side-by-side with the national IMO selection competition in Taiwan. Listed below are just a few of them:

- Mathematical Competition for Junior High School Students (Ministry of Education)
- National Mathematical Competition for High School Students (Ministry of Education)
- International Mathematical Tournament of the Towns (Chiuchang Mathematical Education Foundation)
- Taiwan Regions Mathematics League (Chiu-Chiu Education Foundation)
- American Mathematics Competitions (Chiu-Chiu Education Foundation)

**Appendix A. Lists of High Schools with Gifted Class in Math and Science**

<table>
<thead>
<tr>
<th>Region</th>
<th>School Name</th>
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<tbody>
<tr>
<td>Taipei City</td>
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<tr>
<td></td>
<td>● The First Girl Highs School</td>
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<tr>
<td></td>
<td>● Taipei Jingmei Girls High School</td>
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<tr>
<td></td>
<td>● Taipei Municipal Jianguo High School</td>
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<tr>
<td></td>
<td>● The Affiliated Senior High School of National Taiwan Normal University</td>
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<td>● National Yang Ming Senior High School</td>
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<td>● Wego Private Senior High School</td>
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<td>Northern Taiwan</td>
<td>● Pan-Chiao Senior High School</td>
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<td></td>
<td>● National Wu-Ling Senior High School</td>
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<td></td>
<td>● National Experimental High School at Hsinchu Science Park</td>
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<td></td>
<td>● National Hsinchu Senior High School</td>
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<tr>
<td></td>
<td>● National Hsinchu Girl’s Senior High School</td>
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<tr>
<td>Central Taiwan</td>
<td>● National Taichung First Senior High School</td>
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<td></td>
<td>● National Taichung Girl’s Senior High School</td>
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<td>● Taichung Shi Yuan Senior High School</td>
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<td></td>
<td>● National Taichung Second Senior High School</td>
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<td>● Taichung Municipal Chungming Senior High School</td>
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<td>● Taichung Municipal Hui-Wen High School</td>
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<td></td>
<td>● National Taichung Wen-Hua Senior High School</td>
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Appendix B. Results of Team Taiwan in IMO

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<tr>
<th>Year</th>
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<th>Silver</th>
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NURTURING THE GIFTED STUDENTS IN THE PHILIPPINES

Lecturer: FILMA G. BRAWNER
Director, Science Education Institute
24 July 2012 in Taiwan

In the past 20 years or so, there has been occurring a major direction in the Philippine educational system, and I believe, also those of other countries, which is putting increasing emphasis on the development of scientific and mathematical skills of our students, particularly at the basic education level. The importance of developing scientific and mathematical skills among our student population cannot be more than emphasized. The main reason for doing so goes back to the importance of these skills in solving day to day problems. These skills definitely are tools to enable people to survive within the purview of their homes, communities and even outside of their communities. Perhaps students may not be aware of this far reaching advantage and effects, and may only be focusing their energies at solving algebraic or word problems. But there should be need for them to appreciate and apply what they have learned from these classroom exercises and opportunities in solving day-to-day problems or using mathematical models to develop new products.

Educators definitely have a critical role in nurturing the mathematical talents and skills of the youth under their care. In referring to educators, they may be the different personalities engaged or actively involved in nurturing and developing our students’ mathematical skills. Educators may be those at the policy level; those at the administrative and supervisory level, and those at the classroom level. All of them, however, provide the opportunities for students to develop their mathematics skills. Those at the policy level make sure that opportunities for nurturance are provided, while those in the classroom level, i.e., the teachers, translate these opportunities into their teaching methodologies, and the classroom activities they provide their students. All in all, nurturing the mathematical skills and talents of our students should be a multi-pronged approach, mainly played by educators. It should not be forgotten, though, that provision of opportunities for nurturing at home as well as parental support are also important.

In the Philippine scene, there is now a major educational reform, which is increasing the length of basic education from 10 to 12 years. Correspondingly, efforts are being expended to develop further and deepen the existing curriculum in the subject areas, including mathematics and science. Because of the wide variations in student characteristics and circumstances, it is hoped and expected that after 12 years of basic education, they would have developed more fully the skills needed either to take on employment, or pursue studies for a profession.

As students go through the basic education, it is important that nurturing efforts in
developing their skills, particularly in science and mathematics, should be put in place. It is expected that when nurturing efforts are stepped up, we would be able to produce human resources who can compete with their counterparts.

In the Philippines, the set-up is such that other government agencies and the private sector support the efforts of the Department of Education in its mandate of providing quality basic education. One of such government agency, is the Department of Science and Technology, which looks into helping develop and upgrade science and mathematics education.

Particularly, the Science Education Institute (SEI) of the Department of Science and Technology has initiated programs and projects that would create awareness among the students on career opportunities in the various fields of science and mathematics. These programs include career planning lectures and experiences for high school students. Thus, such advocacy would enable the students to see and appreciate the mathematics value in society and ultimately develop their interests in taking science and mathematics. Mathematics and science camps are also organized for students to experience directly science and math outside of the confines of their classrooms.

The DOST-SEI offers incoming college freshmen a range of scholarships if they plan to pursue a degree in engineering, mathematics and science and other related courses. SEI also supports master and doctoral degree scholarships in basic and applied science and engineering to those who plan to take more specialized degrees. Through such program, a greater number of competent technical manpower would be possibly developed that would help to address the emerging challenges brought about by globalization.

Aside from scholarship grants, SEI also gives full support in the conduct of science and math forums and local and international science and mathematics competitions like the Philippine Mathematics Olympiad (PMO), International Mathematics Olympiad (IMO), and Australian Mathematics Competition (AMC). Such international competitions would give a greater opportunity for young Filipino students to show their competitiveness particularly in the field of science and mathematics.

The Institute likewise promotes science and mathematics through related activities which make use of mathematics, like that of robotics. Students are trained particularly in the field of robotics and programming which may help enhance their technical skills and deepen their scientific and mathematical knowledge that in a way would lead us to produce pool of future scientists, mathematicians and engineers.

DOST desires to continuously make its contribution to mathematics education by coming up with innovative programs. It is hoped that with better developed mathematics skills, our students become better problem-solvers, logical thinkers, planners and organizers.
High School Mathematical Competitions in Russia Nowadays

Lecturer: Fedorenko Igor
Krasnodar, RUSSIA, Bernoulli Center

1. **The list of main math competitions:**
   - All-Russian national olympiad, holds annually by the Russian Ministry of education in traditional form, includind four stages (specific exceptions are 8th grade and St. Petersburg and Moscow olympiads);
   - International Mathematical Tournament of Towns, covering almost all regions of Russia and many foreign countries (Bulgaria, Canada, USA, Iran, Serbia, Ukraine, Belarus, Germany, Lithuania, Kazakhstan, Israel etc), totally about 200 towns all over the world particiate;
   - Tournaments of «mathematical battles» (Festival of Young Mathematicians, Kolmogorov Cup, Ural Tournament) – team mathematical competitions;
   - Igor Fedorovich Sharygin memorial Geometry Olympiad, holds in two stages (preliminary correspondence stage and final internal stage), some foreign students (from Ukraine, Romania, Moldova, Turkey, Kazakhstan, Armenia, India, Vietnam etc) also take part;
   - Festival «GOLDEN FLEECE» - six days long olympiad marathon, elite level oral individual contest.

2. **Problems and peculiarities of math contests organisation**
   - requirements to the age structure of the contest;
   - types of contests: individual and team contests;
   - types of contests: classical (sport-like) and research-like;
   - taking into account the age peculiarities of students: oral olympiads, multiple choice tests;
   - relations with government structures, priority of mathematical society opinion;
   - depending on school programs standards;
   - relations with international systems of mathematical competitions.

3. **Perspectives of mathematical competitions in Russia**
   - broad local provincial initiative against extending gap between students level in capitals (Moscow and St Petersburg) and the Russian province;
   - preparing Decrees by President of Russia V. Putin on supporting the work with gifted high school students and, in particular, on mathematical education development;
   - activisation of the professional mathematical community role, necessity of its precisely formulated politics, its social influence on the system of mathematical olympiads.
GIFT AND RESPONSIBILITY

Lecturer: Alfred K. Lao
Chairman and President, OPTIMA FASHION COLLECTION
Chairman of the Board, ALEXIS MARKETING INC

My wife and I have three daughters who love mathematics. Thanks to the permission of Dr. Simon Chua, President of MTG, we have been accompanying our daughters to mathematics competitions for nine years now; having gone to places as far away as Romania and the United States of America, as near to us as Hong Kong and Taiwan, as exotic and mysterious as India, Nepal and Korea, as ancient and majestic as the various cities of China, as well as to the different lovely places of our own Southeast Asian neighbors, Indonesia, Thailand and Singapore. We are willing to wager that there is no other parent who has logged more frequent flier miles accompanying their children to competitions. Maybe that was the reason why Dr. Simon Chua and 孙文先老师 requested me to share our experiences as parents of mathematically gifted children.

With great power comes great responsibility. This was the sentence popularized in the first Spiderman movie. Similarly, with great gifts come greater responsibilities.

My wife and I were thrust face-to-face with this quandary when our eldest daughter, Carmela was two and a half years old. I still remember it as clearly as if it happened yesterday. We received an urgent call from her nursery teacher requesting us to go to her school asap. Thinking it was some emergency, we dropped all our work and rushed to her school. You can just imagine and feel our palpable relief when the teacher excitedly informed us that Carmela read aloud the whole storybook of Snow White and the Seven Dwarfs from cover-to-cover in front of her classmates!

We knew then we were at a crossroad. To proceed with business as usual would have meant that we were unwilling to assume the greater responsibilities that greater gifts entailed. We are entrepreneurs involved in the bag manufacturing business with plenty of expansion potentials. It was a classical “to be or not to be” situation wherein we had to choose whether to spend more time in our work or to be hands on in the education of our children. Though a difficult decision, we opted to put our own ambitions on hold to help our children pursue theirs.

In a strange twist of fate, our business continued to expand despite our spending less time in it. But that is another story for another time.

When our children were in their formative years of 4 to 8 years old, we would personally tutor them, sit with them while they studied and plowed through their homework, and personally prepared extra lessons for them. My wife took careful note of their progress in English and music while I had primary responsibility over mathematics and science, making worksheets and teaching them math concepts they would not
otherwise have learned in school at their young age. When they reached Grade 3, we enrolled them with Dr. Simon Chua of the MTG Philippines for further mathematics training. Besides mathematics, we advocated holistic development for our children, encouraging them to try different musical instruments. They can all play the piano and the guitar. Carmela is a good drummer. Audrey is a mean guitarist and Czarina plays the violin. They have all tried composing short music though only for church. They are also all varsity pingpong players though none of them can beat me at this point in time. But more importantly, we taught them to dream big and dream bold. We emphasized though that doing so entails a lot of hard work and discipline. We promised them we would support them all the way but the dreams have to be theirs not ours.

I remember that way back 2004 when my daughter, Carmela, was in Grade 4 and after winning one of only 5 Gold medals given at IMSO 2004 (which, by the way, was wonderfully hosted by Indonesia), she suddenly told us that she dreams of one day representing the Philippines in the IMO, the International Mathematical Olympiad, and to end our country's medal drought in that most prestigious of math olympiads. We didn't even know what the IMO was at that time. To support her dreams, we scoured the internet, bought a library of IMO training books from Amazon and introduced her to the Art of Problem Solving website which she still visits to this day.

At the young age of 9, she already decided that she would study at the Massachusetts Institute of Technology in the United States of America when she reaches her collegiate years. This was the dream that surprised and elated us. It was completely beyond our experience and expectations. We definitely dreamt of our children going to the best universities in the Philippines although since my wife and I went to different universities during our undergraduate studies, we couldn’t agree on which Philippine university was the best. We may have dreamt of sending them to NUS, but to send them to what is arguably the top university in the world? Probably never in our wildest dreams. But it was Carmela’s dream to go to MIT that affirmed us in our conviction to further encourage them to dream big and dream bold. She is presently enrolled in her dream school, the Massachusetts Institute of Technology, where she is studying mathematics, having the greatest time in the world, learning under some of the best professors in the world and interacting with equally gifted peers while still maintaining a perfect A+ GPA.

Dreaming bold dreams does not always end in a bed of roses. It is only realistic to expect that some dreams end in failure and disappointment. While our first role as parents is to teach our children to dream big and dream bold, our second role is to be there with them and for them when they experience the inevitable failures and disappointments that comes with dreaming big.

That happened to Carmela in 2008 when she joined the CGMO (China Girls' Math Olympiad) where for the first time in her life she failed to win any medal in a math competition. Luckily, we were there for her. There wasn't even any need for words of encouragement. Just being there for her was all that mattered. That disappointment
ignited her resolve to excel and fueled her to work harder. Daily, she allotted some time to solve math problems. The following year, the habit of dreaming big and bold enabled her to rise like a phoenix from the ashes of defeat to get the Philippines' first and so far only Gold in the entire history of the CGMO.

She did not rest on her laurels for she knew that she had a bigger dream to conquer. She set her sights on fulfilling her IMO dream and never allowed her duties as the school paper's Editor-in-Chief, as the school play's Director, and as a class officer to deprive her of training time in mathematics. She squeezed in training time whenever and wherever she could. Everytime our family would eat out, she brought along her math problems and solved while waiting for the food to come out. She opted to stay home alone whenever our family went out to watch movies. She did not require any prodding to train as hard as she could. Only her dreams sustained her. In 2010, she won the Philippines' first silver medal in the IMO after a drought of 21 years.

But life is not only about winning in mathematics competitions. It is not even only about reaching your most fervent dreams. Our most important task as parents is to make sure our children turn out to be good persons. We need to ensure that they view and practise humility as a real virtue and not as a mere veneer to boost their reputation. They must recognize that real achievement lies not in using others as stepping-stones to the path of success but that success is much sweeter and more sustainable when they can help and inspire others to succeed as well.

My wife and I have tried our best to inculcate these and other virtues in our daughters. Notwithstanding whatever successes they will be able to achieve in the future, we would consider ourselves abject failures if they turn out to be anything less than good persons.
How to Select Good Contestants for Mathematics Competitions

Lecturer: Wen-Hsien SUN(孫文先)
President of Chiu Chang Mathematics Education Foundation

My personal belief:
To select the right people, you just need to select the right problems!

I believe the following topics in math, which are important in everyday life but not taught in school math, can be used to select smart students who are suitable for math research:

- a. Observation skills
- b. Spatial intuition
- c. Dynamic thinking
- d. Logical deduction
- e. Hands-on operation
- f. Creativity

a. Observation skills

- 2003 Semi Final II No.1
  The picture on the right shows the same cat as the picture on the left, except it has been reflected in a mirror. Aside from that, however the two pictures are not exactly alike. There’re ten spots where things are not the same. Can you find all of them?

- 2007 Semi Final II No.2
  The picture on the right shows the same scene as the picture on the left, except it has been reflected in a mirror. Aside from that, however the two pictures are not exactly alike. There’re six spots where things are not the same. Can you find all six?
b. Spatial intuition

2000 Semi Final II No.3
Which photo (numbered 1 to 7) is the negative of the top left photo? Please also indicate where the other photos do not match the left photo. (Small black dots may be due to publishing defects.)

![Image of photos 1 to 7]

b. Spatial intuition

2002 Final No.1
In the five wood pieces shown below, any cube is a unit cube, exactly the same size as all others. Which two pieces can combine to form a perfect 4×4×4 cube? Please explain your reason.

![Image of wood pieces A to E]

c. Dynamic thinking

2004 Semi Final II No.2
When a ball hits a grid point on the side, it is reflected like a light path is reflected by a mirror. When a white ball hits a black ball, the white ball takes up the same position as that black ball, while the black ball moves along the same straight path as the white ball. A black ball must never hit another black ball. See figure A and figure B for an example with one white ball and two black balls which are to be put into pockets numbered 1 and 2:

![Image of grid with arrows and balls]
Try to put the four black balls shown in the following figure into the four different numbered pockets, using the white ball to begin with. Draw the paths of the balls on the figure.

![Diagram of a puzzle with black and white balls and numbered pockets](image)

- **2006 Final II No.4**
  
  UFO was created by Hiroshi Yamamoto and presented by Nob Yoshigahara in February 1998. For each step, a piece travels left, right, up or down towards another piece, stopping in the next to the piece moved towards. Travel is stopped only by another piece and must always be towards another piece. The objective is to move the X piece to the center, which is marked with a *

  - Write down your moves. M→ means piece M travels right; M← means piece M travels left; M↑ means piece M travels up; M↓ means piece M travels down.

- **d. Logical deduction**

- **2001 Final II No.1**

  The order of the following six pictures has been scrambled. Please use the numbers on the pictures to arrange them in a logical sequence.

  ![Six pictures](image)

- **Moon Cake**

  There is a box of moon cakes, with \( n \times n \) squares, and one moon cake per square. After some of the moon cakes are eaten, each row and column contains exactly one moon cake of each flavor (red bean, green bean, white lotus, date, blueberry). The arrows beside the figures denote the first moon cake which can be seen from that
direction. Please fill in the arrangement and flavors of the moon cakes in these three boxes (place a “×” in squares without moon cakes).

**Example**

```
+-----------------+-----------------+-----------------+
|                 |                 |                 |
| Green bean      | Red bean        |                 |
|                 |                 |                 |
| Red bean        | R   G   W       |                 |
|                 | W   R   G       |                 |
|                 | G   R   W       |                 |
|                 | W   G   R       |                 |
|                 | R   W   G       |                 |
|                 |                 | Red bean        |
```

A-1

```
+-----------------+-----------------+-----------------+
|                 |                 |                 |
| Green bean      | Blue berry      | Green bean      |
|                 |                 |                 |
| Green bean      |                 | Red bean        |
|                 |                 |                 |
| White lotus     |                 | Red bean        |
|                 |                 |                 |
| Green bean      |                 | Blue berry      |
|                 |                 |                 |
| Red bean        | Red bean        | White lotus     |
|                 |                 |                 |
```

e. **Hands-on operation**

- 2005 Semi-Finals No.3

Kenneth has a set of Tangrams, and one night he arranges them in the square configuration shown. The second morning he discovers the configuration is now in the
shape of the right figure below. His intuition said that his brother must have lost a piece, but after he looked at the figure carefully again he realized he was wrong. How did his brother arrange the Tangrams into the shape of the right figure?

![Tangram Diagram](image1)

- **2000 Finals II No.2**
  Dissect the figure in the diagram below into two congruent pieces. (You may only cut along the dashed lines.)

![Figure Diagram](image2)

- **2003 Semi Final II No.4**
  We have a carpet of size 7×10 square meters, and the shaded portion in its center decayed because of a heavy cabinet placed there. Please cut the undecayed portion of the carpet into exactly two pieces which can be reassembled to form a carpet of size 8×8 square meters.

![Carpet Diagram](image3)

- **2012 Semi Final II No.4**
  “Pentiamonds” are figures constructed through adjoining five equilateral triangles by their sides. There are four such figures:

![Pentiamonds](image4)

By adjoining two of these figures, both of which may be flipped or rotated, by the
sides of the small triangles, please create a shape with axial symmetry. (Find as many such shapes as you can, and draw the figures that form it in the below triangular net. There are at least 5 such shapes.)

- 2012 Final II No.4
  “Pentiamonds” are figures constructed through adjoining five equilateral triangles by their sides. There are four such figures:

  ![Pentiamonds](image)

  By adjoining three of these figures, all of which may be flipped or rotated, by the sides of the small triangles, please create a shape with axial symmetry. (Find as many such shapes as you can, and draw the figures that form it in the below triangular net. There are at least 21 such shapes.)

**f. Creativity**

- 2007 Final II No.2
  The pictures show what happened on the day that no one could see the object on which each was riding. What’s invisible in each picture?

![Picture](image)

**The Taiwan Experience in Math Competitions**

- After I have selected these smart students, I specify some books and problem sets related to math competitions for them to self-study and discuss.
- Taiwan’s students have performed well at math competitions for many years, and had great achievements afterwards in math research.
Training for Mathematics Competitions in Taiwan: My Views

Lecturer: Prof. Yeong-Nan Yeh
Research Fellow of Institute of Mathematics, Academia Sinica, Taiwan

My association with the International Mathematical Olympiad (IMO) dated back to 1993, when I was involved with the setting of problems for selection of our national team and the training of the team. Over the past two decades, I have been an observer, leader and head of our national team. I was the Chairman of the Problems Committee when Taiwan hosted the IMO in 1998. Over this period, I solved thousands of problems used for training our national team and set some problems for the IMO and the Asian Pacific Mathematical Olympiad. I have also been involved in a number of other mathematics competitions including some organized by the government:

1. Final Contest of the National High School Competition in Mathematical Abilities
2. Selection Contest for the National Mathematical Olympiad Team
3. Asian Pacific Mathematical Olympiad
4. Team Contest of Intercity Youth Mathematics Competition

and some organized by non-government bodies:

1. Tournament of the Towns (Chiu Chang Mathematics Education Foundation)
2. Taiwan Regional Mathematics League (99 Cultural and Educational Foundation)

Mr Sun requested me to talk about mathematics education. As I have never published in mathematics education and am not familiar with its theory, I will talk about my experience.

Mathematical Olympiads target at students with strong interest in mathematics to expose the nature of mathematics, discover their potential, cultivate rigor, flexibility and persistence in their mathematical ability. Recently, our public examination questions in mathematics are mostly very easy. One needs only substitute values into formulas and calculate to get the answers. So one must obtain near perfect scores in order to get into competitive universities. Public examination influences teaching and learning. Students focus on how to obtain a correct answer in a few minutes and skip problems requiring in-depth understanding. Quality of education declines under such atmosphere. As the medical schools are the top preference of our parents, entering them require perfect scores in every subject at the entrance examination. To achieve this, students are required to do a lot of tests everyday, which destroy students’ ability in thinking.
independently. Many students are not permitted by their parents to participate in our selection for the national team so the students can focus on the university entrance examinations.

At the end of every year, the National High School Competition in Mathematical Abilities is held. Problems in this competition cover function, equation, inequality, sequence, complex number, trigonometric function, elementary enumeration in discrete mathematics, introductory probability and statistics, elementary calculus, analytic geometry, solid geometry and elementary vector. The problems in this competition are much easier than those at the IMO. Every time when I started my training sessions with students I asked who had done past problems of this competition. Only students from one school had. Similar responses were received at the training sessions for selection of IMO national team when asked about experience with past IMO problems. The situation is worse with girls. Their teachers are not interested in helping students participate in mathematics competitions. This widens the gap between the performance of boys and girls in mathematics competitions at the high school level. On the other hand, there are less and less professors helping mathematics competition, particularly when compared with a decade ago. This brings difficulty to the training of the national team.

Problems 3 and 6 of the IMO are always a headache to our national team with few marks scored in these two problems. Students are scared of them and focus their preparation on the remaining ones.

When students participate in mathematics competitions and cannot solve a problem, there are different scenarios based on my experience from training them:

1. The problem cannot be solved during the contest but can be solved when the bell rings – this means that our mathematical knowledge in our mind have not been properly organized that we cannot link it with method for solving the problem.
2. The problem cannot be solved during the contest but can be solved when the solution is seen – this means that our strategies for problem-solving are not good enough and we are not aggressive enough in analyzing problems so as to understand clearly the mathematical knowledge in our mind.
3. The problem cannot be solved during the contest nor after seeing the solution but can be solved after explanation by someone – this means that we do not understand thoroughly the mathematical knowledge in our mind with many facts being accepted without understanding.
4. The problem cannot be solved during the contest nor after seeing the solution and even after explanation by someone – this means that our knowledge of mathematics is not enough and with much room for improvement.
Good nutrition is the cornerstone for survival, health and development. Well-nourished children perform better in school, grow into healthy adults and in turn give their children a better start in life. Well-nourished women face fewer risks during pregnancy and childbirth, and their children set off on firmer developmental paths, both physically and mentally. Malnutrition—undernutrition and overnutrition is one of the most serious public health challenges of the 21st century. An alarming number of studies report that overnutrition and the resulting obesity are a growing health problem for children in industrialized nations and even some developing ones. The problem is global and is steadily affecting many low- and middle-income countries, particularly in urban settings. The prevalence has increased at an alarming rate. Globally, in 2010 the number of overweight children under the age of five, is estimated to be over 42 million. Close to 35 million of these are living in developing countries. Obese children and adolescents suffer from both short-term and long-term health consequences. Overweight and obese children are likely to stay obese into adulthood and more likely to develop noncommunicable diseases like diabetes and cardiovascular diseases at a younger age. At least 2.6 million people each year die as a result of being overweight or obese. Furthermore, a lack of calories and nutrients—or undernutrition—can worsen the effects of infectious disease, and thereby causes half of all child deaths worldwide. Undernourished children have lowered resistance to infection and are more likely to die from common childhood ailments like diarrheal diseases and respiratory infections. Frequent illness saps the nutritional status of those who survive, locking them into a vicious cycle of recurring sickness and faltering growth. Globally, more than one third of child deaths are attributable to undernutrition. Many low- and middle-income countries are now facing a "double burden" of disease: as they continue to struggle with the problems of infectious diseases and undernutrition; at the same time they are experiencing a rapid increase in risk factors of NCDs such as obesity and overweight, particularly in urban settings. It is not uncommon to find under-nutrition and obesity existing side-by-side within the same country, the same community and even within the same household in these settings.

WHO recognizes that the increasing prevalence of childhood obesity results from changes in society. The problem is societal and therefore it demands a population-based multisectoral, multi-disciplinary, and culturally relevant approach.

Unlike most adults, children and adolescents cannot choose the environment in which they live or the food they eat. They also have a limited ability to understand the long-term consequences of their behavior. They therefore require special attention when
fighting the obesity epidemic. Malnutrition—either undernutrition or obesity, as well as their related diseases, are largely preventable. Prevention of these nutrition problems therefore needs high priority. Currently there is no dietary recommendation of global utility available for children and adolescents. However, individuals and populations are advised to: (1) increase the consumption of fruit and vegetables, as well as legumes, whole grains and nuts; (2) limit the energy intake from total fats and shift fat consumption away from saturated fats to unsaturated fats; (3) limit the intake of sugars. (4) be physically active - accumulate at least 60 minutes of regular, moderate- to vigorous-intensity activity each day that is developmentally appropriate. Basically, balanced diet and increased physical activity are two principal components to keep health for all children and adults.
Anti-bullying and Legal Education for Gifted Children

Apollo Chen
National Central University, Taiwan

Introduction

Bullying is often seen as an inevitable part of school-yard culture and receiving much more attention than it did a decade ago. In a study of more than 15,000 sixth- through tenth-graders class in the U.S., “30 percent of the students reported bullying others, being the target of bullies, or both.” Gifted and talented students have experience higher incidences of bullying.

What is Bullying

Bullying is usually defined as “a student is being bullied or victimized when he or she is exposed, repeatedly and over time, to negative actions on the part of one or more other students.” Such negative actions include intentionally inflicting, or attempting to inflict, injury or discomfort upon another. These behaviors can be carried out physically (e.g., hitting, kicking, pushing, choking), verbally (e.g., by calling names, threatening, taunting, malicious teasing, spreading nasty rumors), or in other ways, such as making faces or obscene gestures, or intentional exclusion from a group.

Bullying and Gifted Students

In 2006, Jean Peterson at Purdue University found that two-thirds of gifted students encounter bullying by the eighth grade. Jean Peterson also stated that being labeled “different” increases the chance that a gifted child will be bullied—especially if the perpetrator assumes the “gifted” status means an attitude of self-importance. Based on the 2005 survey of 432 gifted students and interviews of 55:

- Victims identified as “gifted” think that what causes bullying is beyond their control—yet they may feel compelled to take responsibility for fixing it themselves, instead of asking for help.
- Just one experience with non-physical bullying can be traumatic, with long-lasting impact.
- Despite the “gifted” label, being bullied can contribute to self-doubt and low self-esteem. “I just felt so bad. My mind was telling me, ‘You’re worthless.’”
- Coping strategies improve with age.
- When children take action, have support, make changes, or the bullying somehow stops (by family relocation, for example), they have higher self-esteem.
- Gifted bullies are able to change their behavior.
- Intelligence helps some gifted kids cope with victimization—they tend to try to “make sense” of the bullying behavior.
- Gifted victims try to avoid “mistakes” and to “be better” in order to avoid being bullied.
- “Not being known” puts kids at risk. When better acquainted, bullies and bullied sometimes can become friends.

Laws related bullying in Taiwan

Legal education is one part of anti-bullying policies and is well-discussion. The legal liabilities of a bullying behavior in Taiwan are as follows:

1. **Teachers’ obligations and liabilities:**

<table>
<thead>
<tr>
<th>OBLIGATIONS</th>
<th>LIABILITIES</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Subparagraph 2 of Article 49 of Children and Youth Welfare Act states that, no one shall treat the children and youth with Physical and mental mistreatment.</strong></td>
<td>Article 100 of Children and Youth Welfare Act states that, any one who violates the provisions of paragraph 1, article 53 without any rationale ground may be fined with a penalty of more than New Taiwan Dollars Six Thousand (NT$6,000) less than New Taiwan Dollars Thirty Thousand (NT$30,000).</td>
</tr>
<tr>
<td><strong>Subparagraph 2 of Article 53 of Children and Youth Welfare Act states that, In case the educationalists implementing children and youth welfare learn that the children or youth fall in Physical and mental mistreatment, they should communicate immediately with the municipal and county (city) authorities not later than twenty-four (24) hours.</strong></td>
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</tbody>
</table>

2. **Bully’s liabilities:**

<table>
<thead>
<tr>
<th>LIABILITIES</th>
<th>ACTION</th>
<th>LAW</th>
<th>NOTE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Criminal Liabilities</td>
<td>Offenses of Causing Injury</td>
<td>Article 277 of Criminal Code of the Republic of China states that, a person who causes injury to another shall be sentenced to imprisonment for not more than three years, short-term imprisonment, or a fine of not more</td>
<td>Please noted that according to Article 18 of Criminal Code of the Republic of China, An offense</td>
</tr>
<tr>
<td>LIABILITIES</td>
<td>ACTION</td>
<td>LAW</td>
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<td>than one thousand yuan. If death results from the commission of an offense specified in the preceding paragraph, the offender shall be sentenced to life imprisonment or imprisonment for not less than seven years; if serious physical injury results, the offender shall be sentenced to imprisonment for not less than three years but not more than ten years.</td>
<td>Article 278 of Criminal Code of the Republic of China states that, a person who causes serious physical injury to another shall be sentenced to imprisonment for not less than five years but not more than twelve years. If death results from the commission of an offense specified in the preceding paragraph, the offender shall be sentenced to life imprisonment or imprisonment for not less than seven years.</td>
<td>committed by a person who is under fourteen years of age is not punishable. Punishment may be reduced for an offense committed by a person more than the age of fourteen but under the age of eighteen. And according to Juvenile Delinquency Act, the juvenile court may pronounce a ruling of the following protective measures when hearing a case: 1. To pronounce a warning and may order holiday consulting; 2. To send a juvenile under probation and supervision and may order labor services; 3. To send a juvenile to a proper welfare or cultivation institute; 4. To send a juvenile to a correction institute for corrective education.</td>
</tr>
<tr>
<td>Offenses Against Freedom</td>
<td>Article 304 of Criminal Code of the Republic of China states that, a person who by violence or threats causes another to do a thing which he has no obligation to do or who prevents another from doing a thing that he has the right to do shall be sentenced to imprisonment for not more than three years, short-term imprisonment, or a fine or not more than three hundred yuan. An attempt to commit an offense specified in the preceding paragraph is punishable.</td>
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<tr>
<td>Threaten</td>
<td>Article 305 of Criminal Code of the Republic of China states that, a person who threatens to cause injury to the life, body, freedom, reputation, or</td>
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<tr>
<td>LIABILITY</td>
<td>ACTION</td>
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<td>property of another and thereby endangers his safety shall be sentenced to imprisonment for not more than two years, short-term imprisonment, or a fine of not more than three hundred yuan.</td>
<td>Article 346 of Criminal Code of the Republic of China states that, a person who by intimidation causes another to deliver over a thing belonging to him or to a third person for purpose to exercise unlawful control over it it for himself or for a fourth person shall be sentenced to imprisonment for not less than six months but not more than five years; in addition thereto, a fine of not more than one thousand yuan may be imposed. A person who by the means specified in the preceding paragraph takes an illegal benefit in property for him or a third person shall be subject to the same punishment. An attempt to commit an offense specified in one of the two preceding paragraphs is punishable.</td>
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<tr>
<td>Insult</td>
<td>Article 309 of Criminal Code of the Republic of China states that, A person who publicly insults another shall be sentenced to short-term imprisonment or a fine of not more than three hundred yuan. A person who by violence commits an offense specified in the preceding paragraph shall be sentenced to imprisonment for not more than one year, short-term imprisonment, or a fine of not more than five hundred yuan.</td>
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<tr>
<td>Offenses Against</td>
<td>Article 310 of Criminal Code of the Republic of China states that, A person</td>
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<tr>
<td>LIABILITIES</td>
<td>ACTION</td>
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<tr>
<td>Reputation</td>
<td>who points out or disseminates a fact which will injure the reputation of another for purpose that it be communicated to the public commits the offense of slander and shall be sentenced to imprisonment for not more than one year, short-term imprisonment, or a fine of not more than five hundred yuan. A person who by circulating a writing or drawing commits an offense specified in the preceding paragraph shall be sentenced to imprisonment for not more than two years, short-term imprisonment, or a fine of not more than one thousand yuan. A person who can prove the truth of the defamatory fact shall not be punished for the offense of defamation unless the fact concerns private life and is of no public concern.</td>
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</tr>
<tr>
<td>Civil Liabilities</td>
<td>Torts</td>
<td>According to Article 184 of Civil Code, a person who, intentionally or negligently, has wrongfully damaged the rights of another is bound to compensate him for any injury arising therefrom. The same rule shall be applied when the injury is done intentionally in a manner against the rules of morals.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>mental anguish</td>
<td>According to Article 195 of Civil Code, a person has wrongfully damaged to the body, health, reputation, liberty, credit, privacy or chastity of another, or to another's personality in a severe way, the injured person may claim a reasonable compensation in money even if such injury is not a purely pecuniary loss. If it was reputation that has been damaged, the injured</td>
<td></td>
</tr>
</tbody>
</table>
person may also claim the taking of proper measures for the rehabilitation of his reputation.

According to Article 97 of Children and Youth Welfare Act, any one who violates the provisions of physical and mental mistreatment may be fined with a penalty of more than New Taiwan Dollars Sixty Thousand (NT$60,000) less than New Taiwan Dollars Three Hundred Thousand (NT$300,000). The name of the responsible person of the business will be made known to the public.

Please noted that according to Article 9 of Administrative Penalty Act, an act committed by a person who has not reached the age of fourteen years is not punishable. Penalty may be reduced for an act committed by a person who is fourteen years of age or older but has not reached the age of eighteen years.

3. Bully’s parents or guardian’s liabilities:
In Taiwan, the minor under seventh year of age has no capacity to make juridical acts, if the minor is over seven years of age, he or she has a limited capacity to make juridical acts. So, if the bully, who has no capacity or limited in capacity to make juridical acts, has wrongfully damaged the rights of another, shall be jointly liable with his guardian for any injury arising therefrom if he is capable of discernment at the time of committing such an act. If he or she is incapable of discernment at the time of committing the act, his guardian alone shall be liable for such injury. In the case of the preceding paragraph, the guardian is not liable if there is no negligence in his duty of supervision, or if the injury would have been occasioned notwithstanding the exercise of reasonable supervision. If compensation cannot be obtained according to the provisions of the preceding two paragraphs, the court may, on the application of the injured person, take the financial conditions among the tortfeasors, the guardian and the injured person into
consideration, and order the tortfeasors or his guardian to compensate for a part or the whole of the injury. (Article 13,187 of Civil Code)

Conclusion

The purpose of law can be divided into two ways, one is prevent illegal behavior in advance, and the other is punish afterwards. The advance prevention is through education of the rule of law to creating an environment of anti-bullying, and makes bully to understand their behavior may bear the legal(criminal or civil) liabilities, so that they may feel pressures and avoid bullying others. If a bully behavior has been done, we could proceed against the bully and his or her parents or guardian through legal procedures in order to prevent future offenses.

However, the law is not omnipotent. If the legal education is not widespread or the law is not enforced, the purpose of advance prevention of law won’t be achieved. And in many cases, the afterwards punishment can not fulfill the harm cause by the bully. In fact, the law can do little in a bullying case. So, the government should pay more attention to this problem, and invite professionals from various fields to come up with a comprehensive plan to deal with the anti-bullying issue.

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vi [Link to the document]
A mom’s sharing

Lecturer : Ms. Shoujuan Huang

Like many of your children here, Brian is a mathematically-gifted young man. As a fifth grader in 2007, he won the overall best prize of IMSO in Indonesia. Four years later, he won a silver medal in IMO 2011 in Netherlands. This year, at the time of writing, he is in Argentina attending IMO 2012 again. He is now 16 years old, a very special, very gifted child.

Being Brian’s mom, I was asked about how to raise a super smart kid like him many times. Frankly, I really have no idea, because I didn’t make it happen. Brian is just who he is, and I am just a very ordinary mom.

Yet today, I am here to share something about Brian and my family, a story mixed with joys and sorrows.

Less than two years ago, around one week before Christmas in 2010, everybody in the school was busy working on the once-a-year Christmas concert. That was in the first semester of Brian’s eighth grade. Like always, Brian performed well in most subjects (except PE); and he was very active in school clubs, too. He was the kind of kid that enjoyed school life a lot. And I was there to help decorate the concert; I enjoyed being with the kids, too.

Just when everything proceeded as usual, Brian suddenly started having a mild fever on and off for about 10 days. That was the first warning signal; I learned later that a mild fever lasting for a while is very dangerous. And he felt very tired and became very sleepy all the time. That’s the second warning signal, his routine behavior changed suddenly. But we thought he was just having a cold and didn’t consider anything bigger, until a blood test several days later came out. The test showed that his condition may be worse than what we assumed, and we transferred him to a bigger hospital in Taipei right away.

Brian was diagnosed with Leukemia two days later. The cancer cells had broken down the function of his bone marrow; he was in a very critical condition. (His white blood cell count was lower than 800 (normally thousands), red blood cell count was only 8.2 (normally over 13), and blood platelet count was only 8000 (normally hundreds of thousands). That explained why he is so tired all the time. The doctor had to give him blood transfusion so he wouldn’t faint. And even after 3 to 4 blood transfusions, his blood counts were still very low. I wondered where all of his blood had gone. I had no idea if the doctor was able to make his blood making function work again. We were so horrified. I cried privately for days although I knew I shouldn’t. I was very critical as the one to give Brian courage and hope. I had to hold myself together, I just couldn’t break down.

And we didn’t have time to panic. The doctor promised that more than 70% of cases of children’s leukemia can be cured, and we could only go for it, knowing that the route is by no means easy. The first treatment period - 35 days of chemotherapy - started right away. Every day of the 35 days in the hospital was a fight. Brian was already very sick and weak when he entered the hospital. Yet the chemo went into his body made him even sicker. He had many kinds of side effects from the chemotherapy. He felt extremely tired all day long. (He slept more than 16 hours a day and still felt fatigue.) He couldn’t eat anything sometimes and threw up several times a day. Some days he had severe stomach ache, and other days he had very bad back pain or leg pain. We just didn’t know what was going to happen the next day.

After the first 35 day of hospitalized chemotherapy, his white blood cell count went to the lowest point. And his red blood cell had not gone any higher. Yet the doctor said the first period of treatment was successful and we can go home to take a break before next period of chemo started. The chemotherapy destroyed cancer cells and weakened Brian’s body too. Now Brian had to try his best to eat all of his strength back before next chemo started…
After the first period, Brian still had to go to the hospital on and off for the second, the third, the fourth… endless various shots. And he was still too sick to go back to school, with a very fragile body; he could only be stuck in the hospital and home. He could no longer to hang out with his friends like he did before. He could not eat his favorite food. He felt so bored and depressed. Life for him was just like in a prison.

Then there came the APMO contest in March 2011. With the encouragement of Mr. Sun and some Jiuzhang senior students, Brian decided to attend the competition even though he wasn’t sure if he could make it.

Suddenly; he had a goal to reach. And it didn’t matter what kind of food he could eat, or if he could go outside to play anymore.

He was very determined and he started to study Olympiad math whenever he could. He carried a note book everywhere, he studied math when he was waiting for his turn to receive chemotherapy, when the IV was connected to his body and he had nowhere to go, when he had insomnia because of the side effect of the medicine. When everybody was out to work and school but him, he had nowhere to go but he could study math. It occupied most of his time and the anticipation of going to the selection camp made some minor side effects more tolerable.

Going to the math camp? The doctor thought we were crazy. Brian’s white blood cell count was barely 2000. It’s dangerous for him to be exposed to the public with so many students in a room for hours. We pleaded with the doctor and promised to be very careful and would certainly stop anytime if Brian didn’t feel right.

(Although our doctor didn’t encourage him to go to the math camp after because there were so many students crowded in the classroom, she still gave Brian a lot of help in adjusting his treatment schedule to the following Taiwan IMO contestant competitions.)

Now going to the math selection camp during the breaks in the treatment became his favorite and most important activity.

The side effects were still there, but he had no time to felt sorrow. He tried to eat even though he felt nausea because the math camps were waiting for him to go to. He even tried to do some easy exercise to build up his strength whenever he could.

And I wish I could say that Brian felt much better when he was able to go to the IMO selecting camp. But it was just not what had happened.

The pain, the fatigue, the weakness, many symptoms occurred time to time to block his road to the competition.

He actually thought of giving up several times when the pain attacked. The severe fatigue and the pressure of being a delegate almost overwhelmed him. We struggled many times about going for it or just giving up.

We couldn’t help to imagine all kinds of worst situations happing. What if he suddenly felt nausea during the contest? What if he got fever on the airplane? What if he got infection? What if he had an accident and got hurt? Even a small cut could be very dangerous to a leukemia patient. It was just too risky to let him go abroad out of the doctor’s reach.

How could I send a leukemia patient to an international competition? I regretted secretly putting him and myself in such a dilemma. We could only pray. None of the above happened.

Luckily, Brian’s worst situation happened only before he got on the airplane last year. He couldn’t practice at all but slept for 16 hours a day everyday for 2 weeks before he went abroad for the contest. Then he was just rested enough, leaving all the illness behind, and suddenly, he was ready. He won a silver medal. He was lucky to have such a good result.

This year, the fifty-third IMO held in Argentina last week, with more practice and much will preparation, he finally got a gold medal, as he wished. This is what we couldn’t imagine one and a half years ago.
Letting him go for his dream was not easy, especially when he was not in a good shape. But we just couldn’t take the happiness of attending the math camp out of his life, instead, we tried to let Brian have a goal:” Go for your dream. Do whatever you enjoy in it. Don’t let anything stop it.” Now, being here to share Brian’s story with you, I would say, it would be very difficult without many of our relatives and friends helping out. Only with their help could I focus on accompanying Brian so fully. Because of Brian’s illness we became closer to our family and friends. It’s a blessing in some ways actually. Now Brian still has about 75 weeks to finish his whole chemotherapy. But all in all, Brian’s condition is getting brighter every day. We are very confident that we will survive it and go on. And I thank you for your listening.

**Brian got Gold medal in IMO 2012.**
Mr. Sun and senior IMO contestants visit Brian in the hospital

2007 IMSO Overall Championship
Mathematical problems in competitions and the teaching of mathematics

Lecturer: CHENG Chun Chor Litwin
The Hong Kong Institute of Education

In many mathematics competitions, problems were set up by all participating regions and usually a team to select questions for the competitions. Usually, some adjustment are made on the proposed questions and new problems are added in the list of problems. Some proposed questions could be textbook nature, for example, finding the sum of the cube of the two roots of a quadratic equations; while some could be typical, for example, in a bag that contain 10 yellow ball, 10 black balls, 12 blue balls, at least how many balls should be taken to ensure that you get balls of two different colours, etc. Good questions with long solution may not be suitable in a competition, and geometric questions are not popular. Questions that required proof are usually not include and hand on questions are few.

In this presentation, I would like to draw some examples that are from competitions or from problem solving exercise and discuss their positive effect in the teaching of mathematics. The questions posed here are meant for students exercise.

Some questions are a bit routine but not typical

【Question】Is $\log_{28} 98$ a rational number?

The number is irrational.

Assume that $\log_{28} 98$ is rational. Then there exist relatively prime positive integers $p, q$ with $\log_{28} 98$.

This is equivalent to $28^{p/q} = 98$, from which $28^p = 98^q$, and $2^{2p}7^p = 2^q7^{2q}$.

By the Fundamental Theorem of Arithmetic, it must be that $2p = q$, and $p = 2q$.

This can only be the case if $p = q = 0$, contradicting the choice of $p$ and $q$.

Hence $\log_{28} 98$ is irrational.

【Question】The length of the three sides of a triangle forms a three term arithmetic progression and the lengths of the three altitudes of the triangle also form a three term arithmetic progression. Prove that the triangle is equilateral.

Let the sides of the triangle have lengths $s, s+a, s+2a$, and the attitudes have lengths $h, h+b, h+2b$. Then
\[ 2A = s(h+2b) = (s+a)(h+b) = (s+2a)h \]

(where \( A \) is the area of the triangle)

And

\[ 0 = 2(s+a)(h+b) - s(h+2b) - (s+2a)h = 2ab \]

Hence either \( a \) or \( b \) is zero. So the triangle is equilateral.

The following question appeared in 2008, and due to the property of the answer, (GCD=1), it may not be appeared again in future competitions.

**Question** If we expand and simplify the expression \((1 + \sqrt{2})^{2008}\), we have \((1 + \sqrt{2})^{2008} = a + b\sqrt{2}\), where \( a \) and \( b \) are positive integers. What is the greatest common divisor of \( a \) and \( b \)?

The answer is that the GCD of \( a \) and \( b \) is 1.

**Proof:**

Suppose \((1 + \sqrt{2})^n = a(n) + b(n)\sqrt{2}\), where \( a(n) \) and \( b(n) \) are positive integers.

Then we have

\[
(a(n)+1) + b(n+1)\sqrt{2} = [a(n) + b(n)\sqrt{2}](1 + \sqrt{2}) = [a(n) + 2b(n)] + [a(n) + b(n)]\sqrt{2}
\]

Therefore, \( a(n+1) = a(n) + 2b(n) \) and \( b(n+1) = a(n) + b(n) \).

So, by Euclidean algorithm,

GCD of \( a(n+1) \) and \( b(n+1) \) = GCD of \( a(n) \) and \( b(n) \).

Since \( a(1) = b(1) = 1 \), it follows from mathematical induction that GCD of \( a(n) \) and \( b(n) \) is 1 for all \( n \).

Of course, we can change the question, say \((1 + \sqrt{3})^{2008} = a + b\sqrt{3}\).

The following question is extracted from a Chinese textbook. In a classroom teaching, it is tried in Three variations of the old problems and two solutions for each variations.

**Question** In \( \triangle ABC \), if the three sides \( a, b, c \) satisfy \( a^2 + b^2 + c^2 = ab + bc + ca \), determine the shape of the \( \triangle ABC \).

**Solution 1:**

The original expression is transformed into
\[
2a^2 + 2b^2 + 2c^2 = 2ab + 2bc + 2ca \\
\Rightarrow (a - b)^2 + (b - c)^2 + (c - a)^2 = 0 \\
\Rightarrow a = b, \ b = c, \ c = a.
\]

Hence \(\triangle ABC\) is an equilateral triangle.

**Solution 2:**

\(\therefore AM \geq GM\)

\[a^2 + b^2 \geq 2ab, \ b^2 + c^2 \geq 2bc, \ c^2 + a^2 \geq 2ac.\]

Equality holds when \(a=b, \ b=c, \ c=a\).

So

\[(a^2 + b^2) + (b^2 + c^2) + (c^2 + a^2) \geq 2ab + 2bc + 2ca\]

\[a^2 + b^2 + c^2 \geq ab + bc + ca\]

That is, if \(a^2 + b^2 + c^2 = ab + bc + ca, \ a=b=c\).

【Variation 1】

In \(\triangle ABC\), if \(\sin^2 A + \sin^2 B + \sin^2 C = \sin A \sin B + \sin B \sin C + \sin C \sin A = 0\), determine the shape of the \(\triangle ABC\).

**Solution 1:**

\[(\sin A - \sin B)^2 + (\sin B - \sin C)^2 + (\sin C - \sin A)^2 = 0\]

Hence \(\sin A = \sin B, \ \sin B = \sin C, \ \sin C = \sin A\). That is \(A=B=C\). \(\triangle ABC\) is an equilateral triangle.

**Solution 2:**

\[\sin^2 A + \sin^2 B + \sin^2 C = \sin A \sin B + \sin B \sin C + \sin C \sin A\]

\(\therefore \sin A, \ \sin B, \ \sin C \geq 0 \quad (0^\circ < A, B, C < 180^\circ)\)

Based on above, we have

\[\sin^2 A + \sin^2 B + \sin^2 C \geq \sin A \sin B + \sin B \sin C + \sin C \sin A\]

Hence when

\[\sin^2 A + \sin^2 B + \sin^2 C = \sin A \sin B + \sin B \sin C + \sin C \sin A\]

We have \(\sin A = \sin B = \sin C\).

【Variation 2】

In \(\triangle ABC\), if the three sides \(a, \ b, \ c\) satisfy \(a^4 + b^4 + c^4 = a^2b^2 + b^2c^2 + c^2a^2\), determine the shape of the \(\triangle ABC\).

**Solution 1:**

According to above work, \((a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2 = 0 \Rightarrow a = b = c\).

Hence \(\triangle ABC\) is an equilateral triangle.
Solution 2:
For \( a^4 + b^4 + c^4 = a^2b^2 + b^2c^2 + c^2a^2 \)
\[ \therefore \frac{a^4 + b^4}{2} \geq \sqrt{a^4b^4} = a^2b^2 \]
Based on the above method, \( a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2 \)
Hence when \( a^4 + b^4 + c^4 = a^2b^2 + b^2c^2 + c^2a^2 \), \( a = b = c. \)

【Variation 3】
In \( \triangle ABC \), if the three sides \( a, b, c \) satisfy \( a^3 + b^3 + c^3 = 3abc \), determine the shape of the \( \triangle ABC \).

Solution 1:
The expression in the question can be transformed to
\[ a^3 + b^3 + c^3 - abc - abc - abc = 0 \Rightarrow a(a^2 - bc) + b(b^2 - ac) + c(c^2 - ab) = 0 \]
If \( a^2 - bc = 0 \), \( b^2 - ac = 0 \), \( c^2 - ab = 0 \), then \( a = b = c. \)
hence \( \triangle ABC \) is an equilateral triangle.

Solution 2:
\[ \therefore \frac{a^3 + b^3 + c^3}{3} \geq \left( a^3b^3c^3 \right)^{\frac{1}{3}} = abc \]
Hence \( a^3 + b^3 + c^3 \geq 3abc \), and equality holds when \( a = b = c. \).
Also, \( a^3 + b^3 + c^3 - 3abc = (a + b)^3 + c^3 - 3abc - 3ab(a + b) \)
\[ \therefore x^3 + y^3 = (x + y)(x^2 - xy + y^2) \]
So
\[ (a + b)^3 + c^3 - 3ab(c + a + b) = (a + b + c) \left[ (a + b)^2 - (a + b)c + c^2 - 3ab \right] \]
\[ = (a + b + c) \left( c^2 + a^2 + b^2 - ac - bc - ab \right) \]
Hence when \( a^3 + b^3 + c^3 - 3abc = 0, \)
By \( a + b + c \neq 0 \), we have \( a^2 + b^2 + c^2 - ab - bc - ab = 0 \), that is, \( a = b = c. \).

【Question】We are given a piece of paper. We cut it into 40 or 50 arbitrary pieces, then we choose one of the pieces and cut it into either 40 or 50 pieces. We repeat this process some finite number of times then stop. Can we obtain exactly 1825 pieces in this way? How about 1824?

In number theory:
Suppose \( a \) and \( b \) are positive integers such that \( \text{GCD}(a, b) = 1 \), then for every integer greater than or equal to \( (a - 1)(b - 1) \), there exist Non-Negative integers \( x \) and \( y \) such that
\[ N = ax + by \]
But the equation does not have a solution for \( N = (a - 1)(b - 1) - 1 \).

Suppose we have \( n \) pieces of paper. If we cut 1 piece into 40 (or 50) pieces, then we will have \( n + 39 \) (or \( n + 49 \)) pieces. So after \( x \) cuts into 40 pieces and \( y \) cuts into 50 pieces, we have \( 1 + 39x + 49y \) pieces. Since \((39 - 1)(49 - 1) = 1824\), \( 1 + 39x + 49y = 1825 \) is solvable but \( 1 + 39x + 49y = 1824 \) is not solvable.

For a simpler version, we replace 40 and 50 by 4 and 5.

【Question for teaching】
We are given a piece of paper. We cut it into 4 or 5 arbitrary pieces, then we choose one of the pieces and cut it into either 4 or 5 pieces. We repeat this process some finite number of times then stop. How many pieces can we get after a finite number of steps?

【Question】Is there a power of 2 for which the digits can be rearranged to get a different power of 2?

Assume that it could. Then there are two numbers \( a \) and \( b \), with the same digits and such that each is a power of two.

Assume that \( a < b \). Then \( b = 2a \), \( b = 4a \), or \( b = 8a \) (if \( b = 2^k a \), with \( k \geq 4 \), then \( b \) has more digits than \( a \).

Also, when both \( a \) and \( b \) have the same digits sum, \((b - a)\) must be a multiple of 9.

However, none of the following \( b - a = 2a - a = a, b - a = 4a - a = 3a, b - a = 8a - a = 7a \), are multiple of 9.

Hence, there is no such number.

The number 2 could be changed to other numbers.

【Question for mathematical thinking】
As shown in the diagram, \( \triangle ABC \) is a right angle triangle, \( \angle B = 90^\circ \). Point \( O \) is the centre, with \( OB \) as radius, Line \( AC \) touches the circle at \( D \). Base on this diagram to devise a new question. You are not allowed to add any new points, but you may impose some new conditions.

\( Q1: \) Prove that \( DE \parallel OC \).

\( Q2: \) Prove that \( \frac{CB}{BO} = \frac{AD}{AE} \).
Q3: If $AE=1$, and $\cos A = \frac{4}{5}$, find the area of the circle.

Q4: If $AD=2$, and $AE=1$, find the length of the diameter of the circle, the length $CB$ and the value of $\sin \frac{\angle ACB}{2}$. Prove that the value of $S_{\triangle AOD}$ and $S_{\triangle BCD}$ are the roots of the equation $10x^2 - 51x + 54 = 0$.

Q5: If $AD=2AE$, prove that $BE=3AE$, $BE=BC$.

Q6: If $AD:DC=2:1$, and $DE + BD = 4 + 2\sqrt{2}$, find $S_{\triangle ABC}$, the area of triangle $ABC$.

Q7: If $AD=4$, $CD=6$, find the value of $\tan \angle ADE$.

Q8: If $D$ is the mid point of $AC$, prove that the length of $DE$ is equal to the radius of the circle.

Q9: If $BE=BC$, and the radius of the circle is $r$, find the length of $AE$, and prove that $BE=3AE$, $AD=2AE$.

Q10: If the radius of the circle is $r$, find the value of $ED \cdot OC$. And if $ED + OC = \frac{9}{2}r$, find the length of $CD$.

(The presenter would like to thank Prof Poon Yiu Tung for some of the suggested questions and solutions).
Tessellations with Regular Polygons

Lecturer: Professor Andy Liu
Professor, Department of Mathematical and Statistical Sciences University of Alberta, Canada.

Section 1. Regular Polygons

Let \( n \) be an integer greater than or equal to 3. A regular \( n \)-gon is a polygon with \( n \) sides such that all sides have the same length and all angles have the same measure. The regular 3-gon is called the equilateral triangle. The sum of the measures of its three angles is \( 180^\circ \). Hence the measure of each of its angles is \( 180^\circ / 3 = 60^\circ \).

The regular 4-gon is called the square. It may be dissected into two triangles by either of its diagonals. It follows that the sum of the measures of its four angles is \( 2 \times 180^\circ = 360^\circ \). Hence the measure of each of its angles is \( 360^\circ / 4 = 90^\circ \).

In general, the regular \( n \)-gon may be dissected into \( n - 2 \) triangles by its diagonals from any of its vertices. It follows that the sum of the measures of its \( n \) angles is \( (n - 2)180^\circ \). Hence the measure of each of its angles is \( \frac{n - 2}{n} 180^\circ \).

For which values of \( n \) is the measure of the angles an integer? Let this integer be \( k \). Then \( \frac{n - 2}{n} 180^\circ = k^\circ \) so that \( n(180 - k)^\circ = 360^\circ \). It follows that \( n \) must be a divisor of 360. The chart below gives the values of these measures for each divisor of 360 other than 1 and 2.

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angle</td>
<td>—</td>
<td>—</td>
<td>60°</td>
<td>90°</td>
<td>108°</td>
<td>120°</td>
<td>135°</td>
<td>140°</td>
</tr>
<tr>
<td>( n )</td>
<td>10</td>
<td>12</td>
<td>15</td>
<td>18</td>
<td>20</td>
<td>24</td>
<td>30</td>
<td>36</td>
</tr>
<tr>
<td>Angle</td>
<td>144°</td>
<td>150°</td>
<td>156°</td>
<td>160°</td>
<td>162°</td>
<td>165°</td>
<td>168°</td>
<td>170°</td>
</tr>
<tr>
<td>( n )</td>
<td>40</td>
<td>45</td>
<td>60</td>
<td>72</td>
<td>90</td>
<td>120</td>
<td>180</td>
<td>360</td>
</tr>
<tr>
<td>Angle</td>
<td>171°</td>
<td>172°</td>
<td>174°</td>
<td>175°</td>
<td>176°</td>
<td>177°</td>
<td>178°</td>
<td>179°</td>
</tr>
</tbody>
</table>

Note that \( 360^\circ / 60^\circ = 6 \), \( 360^\circ / 90^\circ = 4 \) and \( 360^\circ / 120^\circ = 3 \). These are the only cases with integral quotients. Thus we can only surround a point with the same kind of regular polygons, without overlap, by using 6 equilateral triangles, 4 squares or 3 regular hexagons, as shown in Figure 1.

![Figure 1](image)

We shall describe a vertex by the numbers of sides of the polygons surrounding it. Thus the three vertices above may be described as (3,3,3,3,3), (4,4,4) and (6,6,6). Such a sequence is called a vertex sequence. Are there other combinations of polygons which can surround a point? We seek a set of angles whose total measure is \( 360^\circ \). We can find some of them by inspecting the values in the chart above, but let us use a more system approach.

We first consider combinations with three polygons. Let the vertex sequence be \((a, b, c), a \leq b \leq c\). Then \(a^{-2}180^\circ + b^{-2}180^\circ + c^{-2}180^\circ = 360^\circ\), which simplifies to

\[
\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}.
\]  

(1)
Suppose $a = 3$. Then (1) becomes $\frac{1}{b} + \frac{1}{c} = \frac{1}{6}$. Solving for $c$ in terms of $b$, we have $\frac{1}{c} = \frac{b-6}{b}$. Hence $b > 6$. For $b = 7$, $c = 42$. For $b = 8$, $c = 24$. For $b = 9$, $c = 18$. For $b = 10$, $c = 15$. For $b = 11$, $c$ is not an integer. For $b = 12$, $c = 12$. Since $b \leq c$, the only solutions in this case are $(3,7,42)$, $(3,8,24)$, $(3,9,18)$, $(3,10,15)$ and $(3,12,12)$.

Suppose $a = 4$. Then (1) becomes $\frac{1}{b} + \frac{1}{c} = \frac{1}{4}$. Solving for $c$ in terms of $b$, we have $\frac{1}{c} = \frac{b-4}{4b}$ so that $c = \frac{4b}{b-4}$. Hence $b > 4$. For $b = 5$, $c = 20$. For $b = 6$, $c = 12$. For $b = 7$, $c$ is not an integer. For $b = 8$, $c = 8$. Since $b \leq c$, the only solutions in this case are $(4,5,20)$, $(4,6,12)$ and $(4,8,8)$.

Suppose $a = 5$. Then (1) becomes $\frac{1}{b} + \frac{1}{c} = \frac{3}{10}$. Solving for $c$ in terms of $b$, we have $\frac{1}{c} = \frac{10b-3b}{3b}$ so that $c = \frac{10b}{3b-10}$. Recall that $b \geq a = 5$. For $b = 5$, $c = 10$. For $b = 6$ or $7$, $c$ is not an integer. For $b = 8$, we already have $b > c$. Thus the only solution in this case is $(5,5,10)$.

Suppose $a = 6$. For $c \geq 7$, $\frac{1}{a} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{2}{b} + \frac{1}{c} = \frac{29}{12}$, which is a contradiction. Hence $a = b = c = 6$, so that the only solution in this case is $(6,6,6)$.

For $a \geq 7$, $\frac{1}{a} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{4}{7}$, which is a contradiction. Hence there are no further solutions. Note that in the case $(3,7,42)$, neither the regular 7-gon nor the regular 42-gon has angles with integral measures. Thus this combination is not easy to discover just by inspection.

We next consider combinations with four polygons. Let the vertex sequence be a permutation of $(a, b, c, d)$, $a \leq b \leq c \leq d$. As before, we have

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 1. \quad (2)$$

Suppose $a = 3$. Then (2) becomes $\frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{3}{2}$. If $b = 3$ we have $\frac{1}{c} + \frac{1}{d} = \frac{1}{3}$. Solving for $d$ in terms of $c$, we have $\frac{1}{d} = \frac{c-3}{3c}$ so that $d = \frac{3c}{c-3}$. Hence $c > 3$. For $c = 4$, $d = 12$. For $c = 5$, $d$ is not an integer. For $c = 6$, $d = 6$. Since $c \leq d$, the only solutions in this subcase are $(3,3,4,12)$ and $(3,3,6,6)$. If $b = 4$ we have $\frac{1}{c} + \frac{1}{d} = \frac{1}{12}$. Solving for $d$ in terms of $c$, we have $\frac{1}{d} = \frac{c-12}{12c}$ so that $d = \frac{12c}{c-12}$. Recall that $c \geq b = 4$. For $c = 4$, $d = 6$. For $c = 5$, we already have $c > d$. Thys the only solution in this subcase is $(3,4,4,6)$. For $b \geq 5$, $\frac{3}{5} = \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq \frac{3}{5}$, which is a contradiction. Hence there are no further solutions in this case.

Suppose $a = 4$. For $d \geq 5$, $1 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq \frac{3}{4} + \frac{1}{5} = \frac{10}{20}$, which is a contradiction. Hence $a = b = c = d = 4$, so that the only solution in this case is $(4,4,4,4)$.

For $a \geq 5$, $1 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq \frac{4}{5}$, which is a contradiction. Hence there are no further solutions.

Now we consider combinations with five polygons. Let the vertex sequence be a permutation of $(a, b, c, d, e)$, $a \leq b \leq c \leq d \leq e$. As before, we have

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} = \frac{3}{2} \quad (3)$$

For $c \geq 4$, $\frac{3}{2} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} \leq \frac{2}{3} + \frac{3}{4} = \frac{17}{12}$, which is a contradiction. Hence $a = b = c = 3$ so that $\frac{1}{a} + \frac{1}{b} = \frac{1}{2}$. As before, we have $(d,e) = (3,6)$ or $(4,4)$, so that the only solutions in this case are $(3,3,3,3,6)$ and $(3,3,3,4,4)$.

Finally, we consider combinations with six polygons. Let the vertex sequence be a permutation of $(a, b, c, d, e, f)$, $a \leq b \leq c \leq d \leq e \leq f$. As before, we have $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \frac{1}{f} = 2$. For $f \geq 4$,

$$2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \frac{1}{f} = \frac{5}{3} + \frac{1}{4} + \frac{23}{12},$$

which is a contradiction. Hence $a = b = c = d = e = f = 3$, so that the only solution in this case is $(3,3,3,3,3,3)$.

We cannot surround a point with seven or more polygons since the smallest of the angles at this point is at least $60^\circ$ and the sum of these angles will exceed $6 \times 60^\circ = 360^\circ$.

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Several of our solutions give rise to more than one vertex sequence. The combination (3,3,4,12) may be permuted as (3,4,3,12). The combination (3,3,6,6) may be permuted as (3,6,3,6). The combination (3,4,4,6) may be permuted as (3,4,6,4). Finally, the combination (3,3,3,4,4) may be permuted as (3,3,4,3,4). There is a left-handed version and a right-handed version of (3,3,3,3,6), but they are not considered to be different. This brings the total number of vertex sequences to 21.

Section 2. Platonic and Archimedean Tilings

For each vertex sequence, we wish to know if we can tile the entire plane with regular polygons such that every vertex has this vertex sequence. Such a tessellation is said to be semi-regular. If moreover all terms in the vertex sequences are identical, it is said to be regular and are called Platonic tilings. Semi-regular tessellations which are not regular are called Archimedean tilings. They are named after two Greek philosophers.

We consider the 21 possible vertex sequences in three groups.

**Group I.** (3,7,42), (3,8,24), (3,9,18), (3,10,15), (4,5,20) and (5,5,10).

These vertex sequences are all of the form \((a, b, c)\) where \(a\) is odd and \(b \neq c\). For (4,5,20), read (5,4,20). If there is a tessellation in which every vertex has this vertex sequence, we must be able to surround the \(a\)-sided polygon. Its neighbors must be the \(b\)-sided and the \(c\)-sided polygon alternately. However, this is impossible since \(a\) is odd. Figure 2 illustrates the case (5,5,10).

**Group II.** (3,3,4,12), (3,3,6,6), (3,4,3,12), (3,4,4,6).

Here the problem is not local but global. We can indeed surround the largest polygon so that each of its vertices has the same vertex sequence. However, at least one other vertex cannot possibly have that vertex sequence.

Figure 2

Figure 3
Figure 4

Figure 3 illustrates the case (3,3,4,12), Figure 4 illustrates the case (3,3,6,6), Figure 5 illustrates the case (3,4,3,12) and Figure 6 illustrates the case (3,4,4,6),

Figure 5

Figure 6

**Group III.** (3,12,12), (4,6,12), (4,8,8), (6,6,6), (3,4,6,4), (3,6,3,6), (4,4,4,4), (3,3,3,3,6), (3,3,3,4,4), (3,3,3,3,3).

Here we have neither local problem nor global problem, so that there are exactly three Platonic tilings (3,3,3,3,3), (4,4,4,4) and (6,6,6) The other eight are the only Archimedean tilings. It is easy to verify that we have no local problem here, but it is not at all easy to prove that we have no global problem either. We shall use a direct approach by constructing each of the eleven tessellations.

The (3,3,3,3,3) and the (3,6,3,6) tessellations may be constructed with three infinite families of evenly spaced parallel lines forming 60° angles across families. See Figure 7.

Figure 7
The (4,4,4,4) tessellation may be constructed with two infinite families of evenly spaced parallel lines forming 90° angles across families. This tessellation and the (3,3,3,3,3) and (3,6,3,6) tessellations are the three basic tessellations. The (3,3,4,4) tessellation is obtained by taking alternate strips from the basic (3,3,3,3,3) and (4,4,4,4,4) tessellations. See Figure 8.

![Figure 8](image)

The remaining tessellations are obtained from others by the cut and merge method. Figure 9 shows the basic (3,3,3,3,3) tessellation with a set of six equilateral triangles merged into a regular hexagon.

![Figure 9](image)

Figure 10 shows the (6,6,6) and the (3,3,3,3,6) tessellations which may be obtained from the basic (3,3,3,3,3) tessellations by merging various sets of six equilateral triangles. No cutting is required in either case.

![Figure 10](image)

Next, we construct the (4,8,8) tessellation from the basic (4,4,4,4) tessellation. We cut each square tile into five pieces, as shown in Figure 11 on the left, consisting of a regular octagon and four congruent right isosceles triangles. Let the edge length of the square tile be 1 and the length of the hypotenuse of the triangles be \(x\). Then the legs of the triangles have length \(\frac{x}{\sqrt{2}}\). From \(\frac{x}{\sqrt{2}} + x + \frac{x}{\sqrt{2}} = 1\), we have \(x = \sqrt{2} - 1 \approx 0.412\). When we merge the triangles across four square tiles, we obtain the (4,8,8) tessellations, as shown in Figure 11 on the right.
The $(3,12,12)$ tessellation may be constructed from the $(6,6,6)$ tessellation in a similar way. We cut each hexagonal tile into seven pieces, as shown in Figure 12 on the left, consisting of a regular dodecagon and six congruent isosceles triangles with vertical angles $120^\circ$. Let the edge length of the hexagonal tile be 1 and the length of the base of the triangles be $x$. Then the equal sides of the triangles have length $\frac{x}{\sqrt{3}}$. From $\frac{x}{\sqrt{3}} + x + \frac{x}{\sqrt{3}} = 1$, we have $x = 2\sqrt{3} - 3 \approx 0.464$. When we merge the triangles across three hexagonal tiles, we obtain the $(3,12,12)$ tessellations, as shown in Figure 12 on the right.

We now obtain the $(3,4,6,4)$ tessellation from the basic $(3,3,3,3,3)$ tessellation. We cut each triangular tile into seven pieces, as shown in Figure 13 on the left, consisting of an equilateral triangle, three congruent half-squares and three congruent kites with angles $120^\circ$, $90^\circ$, $60^\circ$ and $90^\circ$. Let the edge length of the triangular tile be 1 and the length of the side of the equilateral triangle be $x$. Then the short sides of the kite have length $\frac{x}{2}$ and the long sides $\frac{\sqrt{3}x}{2}$. From $\frac{\sqrt{3}x}{2} + x + \frac{\sqrt{3}x}{2} = 1$, we have $x = \sqrt{3} - 1 \approx 0.366$. When we merge the kites and half-squares across six triangular tiles, we obtain the $(3,4,6,4)$ tessellations, as shown in Figure 13 on the right.

The $(4,6,12)$ tessellation can now be obtained from the $(3,4,6,4)$ tessellation without cutting. Each dodecagon in the new tessellation is obtained by merging one regular hexagon, six squares and six equilateral triangles in the old tessellation, as shown in Figure 14.
The last tessellation, namely (3,3,4,3,4), is the most difficult to get. It is obtained from the basic (4,4,4,4) tessellation with an intermediate step.

We first modify the square tile as shown in Figure 15 on the left. Cut out two isosceles triangles with vertical angles 150°, based on two opposite sides of the square, and attached them to the other two sides. This modified tile can also tile the plane, as shown in Figure 15 on the right.

We now cut each modified tile into six pieces, as shown in Figure 16 on the left, consisting of two congruent equilateral triangles and four congruent right isosceles triangles. When we merge the right isosceles triangles across four modified tiles, we obtain the (3,3,4,3,4) tessellations, as shown in Figure 16 on the right.
Section 3. Uniform Tessellations

An important property of a semi-regular tessellation is the following. Suppose we choose two arbitrary vertices $A$ and $B$. Needless to say, they have the same vertex sequence. We make a transparency of the tessellations. Then we can superimpose the transparency onto the original so that vertex $B$ on the transparency coincides with vertex $A$, and the two tessellations coincide.

A tessellation of the plane with regular polygons which has this property is called a uniform tessellation. Clearly, all semi-regular tessellations are uniform, but the converse is not true. There are uniform tessellations which have $n$ kinds of vertex sequences with $n \geq 1$. Such a uniform tessellation is said to be of order $n$. For the remainder of our discussion, we focus on uniform tessellations of order 2, and refer to them simply as uniform tessellations.

Because of their local problems, we cannot incorporate those vertex sequences in Group I into a uniform tessellation. However, we can do so with those in Group II. In fact, Figures 3 to 6 may be developed into uniform tessellations, shown respectively in Figures 17 to 20.

The uniform tessellation in Figure 17 has two kinds of vertex sequences, $(3,3,4,12)$ from Group II and $(3,3,3,3,3)$ from Group III. It may be obtained from the $(4,6,12)$ tessellation by cutting each hexagon into six equilateral triangles.

The uniform tessellation in Figure 18 has two kinds of vertex sequences, $(3,3,6,6)$ from Group II and $(3,3,3,3,3,3)$ from Group III. It may be obtained from the $(6,6,6)$ tessellation by cutting some hexagons into six equilateral triangles.
The uniform tessellation in Figure 18 has two kinds of vertex sequences, (3,4,3,12) from Group II and (3,12,12) from Group III. It may be obtained from the basic (4,4,4) tessellation by cutting each square tile into a regular dodecagon, four squares and eight triangles with angles $30^\circ$, $60^\circ$ and $90^\circ$. The vertices of the square tile are marked with black dots.

![Figure 18](image)

The uniform tessellation in Figure 19 has two kinds of vertex sequences, (3,4,4,6) from Group II and (3,4,6,4) from Group III. It may be obtained from the basic (3,3,3,3,3) tessellation by cutting each triangular tile into a regular hexagon, nine half-squares, three equilateral triangles and nine kites with angles $120^\circ$, $90^\circ$, $60^\circ$ and $90^\circ$. The vertices of the triangular tile are marked with black dots.

![Figure 19](image)

The uniform tessellation in Figure 20 has two kinds of vertex sequences, (3,3,6,6) from Group II and (3,3,3,3,3) from Group III. It may be obtained from the basic (3,3,3,3,3) tessellation by combining sets of six equilateral triangles into regular hexagons. However, the one on the right may also be obtained by shifting strips of the basic (3,6,3,6) tessellation.

![Figure 20](image)

There are two other uniform tessellations which feature the vertex sequence (3,3,6,6). In the one shown in Figure 21 on the left, the other vertex sequence is (3,3,3,3,6). In the one shown in Figure 21 on the right, the other vertex sequence is (3,6,3,6). Both may be obtained from the basic (3,3,3,3,3) tessellation by combining sets of six equilateral triangles into regular hexagons. However, the one on the right may also be obtained by shifting strips of the basic (3,6,3,6) tessellation.
By taking alternate strips from the basic (4,4,4,4) and (3,6,3,6) tessellations, we can get a uniform tessellation featuring the vertex sequence (3,4,4,6) along with (3,6,3,6). This can be done in two ways, as shown in Figure 22.

Apart from these eight cases, there are twelve other uniform tessellations featuring two kinds of vertex sequences. Here, both are from Group III.

The first four are obtained by using combinations of strips from the basic (3,3,3,3,3) and (4,4,4,4) tessellations, each featuring the additional vertex sequence (3,3,4,4). These are shown in Figures 23 and 24.
Figure 24

Figure 25 shows two different uniform tessellations with the vertex sequences (3,3,3,3,3) and (3,3,3,3,6). Both may be obtained from the basic (3,3,3,3,3) tessellation by merging various sets of six equilateral triangles. No cutting is required in either case.

Figure 25

Alternatively, the second of these two tessellations may be obtained from the basic (3,3,3,3,3) or the (6,6,6) tessellations, by cutting without merging. These constructions are shown in Figure 26.

Figure 26

Two more uniform tessellations may be obtained by cutting without merging. The first starts from the (3,4,6,4) tessellation. Each regular hexagon is cut into six equilateral triangles. The resulting tessellation features the vertex sequences (3,3,3,3,3) and (3,3,4,3,4). This is shown in Figure 27. The
second starts from the (3,12,12) tessellation. Each regular dodecagon is cut into a regular hexagon, six squares and six equilateral triangles. The resulting tessellation features the vertex sequences (3,3,4,3,4) and (3,4,6,4). This is shown in Figure 28.

![Figure 27](image-url)

Figure 27

![Figure 28](image-url)

Figure 28

The next uniform tessellation, starting from the basic (3,3,3,3,3) tessellation, requires both cutting and merging. Each equilateral triangle is cut into four equilateral triangles, six half-squares and three kites with angles 120°, 90°, 60° and 90°. When we merge the half-squares across two tiles and the kites across six tiles, we obtain a tessellation which features the vertex sequences (3,3,3,4,4) and (3,4,6,4). This is shown in Figure 29.

![Figure 29](image-url)

Figure 29
Here is another uniform tessellation which requires both cutting and merging. It starts from the basic (3,6,3,6) tessellation. Each equilateral triangle is cut into an equilateral triangle, three half-squares and three kites with angles 120°, 90°, 60° and 90°. Each regular hexagon is cut into a regular dodecagon, six half-squares and six pentagons each of which is a union of two kites with angles 120°, 90°, 60° and 90°. When we merge the half-squares across a triangular tile and a hexagonal tile, and the kites and pentagons across two triangular tiles and two hexagonal tiles, we obtain a tessellation which features the vertex sequences (3,4,6,4) and (4,6,12). This is shown in Figure 30.

![Figure 30](image)

The next uniform tessellation features the vertex sequences (3,3,3,4,4) and (3,3,4,3,4). It is obtained from the basic (4,4,4,4) tessellation as well, with an intermediate step which is quite difficult.

We first modify the square tile as shown in Figure 33 on the left. Let the side length of the square tile be 1. Cut out two quadrilaterals with parallel bases having length 0.512 and 0.422 respectively, with the longer base along two opposite sides of the square, and both bases placed symmetrically with respect to the square. These are then attached to the other two sides, again placed symmetrically with respect to the square. This modified tile can also tile the plane, as shown in Figure 31 on the right.

![Figure 31](image)

We now cut each modified tile into twelve pieces, as shown in Figure 32 on the left, consisting of two squares, six equilateral triangles and four quarter-squares which are kites with angles 120°, 90°, 60° and 90°. When we merge the quarter-squares across four modified tiles, we obtain the uniform tessellations shown in Figure 32 on the right.
The last uniform tessellation also features the vertex sequences (3,3,4,4) and (3,3,3,4). It is obtained from the basic (4,4,4,4) tessellation with an intermediate step.

We first modify the square tile as shown in Figure 33 on the left. Cut out an isosceles triangle with vertical angle 150°, based on one side of the square, and attached it to the opposite side. Then bisect one of the remaining sides and cut out two isosceles triangles with vertical angles 150°, based on each half of this divided side, and attach them to the opposite side. This modified tile can also tile the plane, as shown in Figure 33 on the right.

We now cut each modified tile into twenty-four pieces, as shown in Figure 34 on the left, consisting of eight squares and sixteen equilateral triangles. The uniform tessellations shown in Figure 34 on the right requires no merging.

That there are only twenty uniform tessellations of order 2 is due to Krötenheerdt [3], an incomplete reference provided in the definitive treatise of the subject by Grünbaum and Sheppard [2]. The proof is along the line of our argument that there are only eleven semi-regular tessellations, but clearly the details must be much more intricate.
Exercises

1. (a) Prove that the sum of measures of the exterior angles of a regular \( n \)-gon is \( 360^\circ \) for any \( n \geq 3 \).
   
   (b) Use (a) to give a different proof that the sum of the measures of the angles of a regular \( n \)-gon is given by \( (n-2)180^\circ \).

2. (a) Find three ways of obtaining the basic (3,6,3,6) tessellation from other tessellations.
   
   (b) Find another way of obtaining the (6,6,6) tessellation from the basic (3,3,3,3,3,3) tessellation.
   
   (c) Find a way of obtaining the (3,4,6,4) tessellation from the (6,6,6) tessellation.

3. Explain why the tessellations in the diagram below are not uniform.

![Diagram](image)

References


Solutions

1. (a) Consider a vector moving along one side of the regular polygon. Let the head of the vector pass through the far vertex of the side and wait until the tail of the vector reaches it. Then pivot around this vertex so that the vector is now in the direction of the adjacent side. It has turned through an exterior angle of the polygon. Continue in this manner until the vector returns to its original position. So it has made a \( 360^\circ \) turn. In the process, it has turned through each exterior angle once. Hence the sum of the exterior angles is equal to \( 360^\circ \).

   (b) The sum of each of the \( n \) pairs of interior angle and exterior angle is \( 180^\circ \). Hence the sum of all pairs is \( n180^\circ \). From (a), the sum of all exterior angles is \( 2 \times 180^\circ \). Hence the sum of all interior angles is \( (n-2)180^\circ \).
2. (a) The basic (3,6,3,6) tessellation may be obtained from the basic (3,3,3,3,3,3) in two ways. First, we combine various sets of six equilateral triangles into regular hexagons. Second, we can cut each equilateral triangle into four equilateral triangles as shown in the diagram below on the left. By merging equilateral triangles across six tiles, we obtain the basic (3,6,3,6) tessellation, as shown in the diagram on the right.

![Diagram of (3,6,3,6) tessellation]

Finally, the basic (3,6,3,6) tessellation may be obtained from the (6,6,6) tessellation by cutting each regular hexagon into a regular hexagon and six congruent isosceles triangles with vertical angles 120°, as shown in the diagram below on the left. When we merge isosceles triangles across three tiles, we obtain the basic (3,6,3,6) tessellation, as shown in the diagram on the right.

![Diagram of (6,6,6) tessellation]

(b) The (6,6,6) tessellation may be obtained from the basic (3,3,3,3,3,3) tessellation by cutting each equilateral triangle into a regular hexagon and three congruent equilateral triangles, as shown in the diagram below on the left. When we merge equilateral triangles across six tiles, we obtain the basic (6,6,6) tessellation, as shown in the diagram on the right.

![Diagram of (3,4,6,4) tessellation]

(c) The (3,4,6,4) tessellation may be obtained from the (6,6,6) tessellation by cutting each regular hexagon into a regular hexagon, six congruent half-squares and six congruent kites with angles 120°, 90°, 60° and 90°, as shown in the diagram below on the left. When we merge half-squares across two tiles and kites across three tiles, we obtain the (3,4,6,4) tessellation, as shown in the diagram on the right.

![Diagram of (3,4,6,4) tessellation]

3. In each tessellation, we can choose two vertices with vertex sequence (3,3,3,3,3,3) such that one has only two neighbors with vertex sequence (3,3,3,4,4) but the other has more. Thus the condition in the definition of a uniform tessellation is violated.