5th Invitational World Youth Mathematics Inter-City Competition

第五屆青少年數學國際城市邀請賽

Individual Contest Time limit: 120 minutes 2004/8/3, Macau

Team: ______________ Contestant No.____________ Score: ________________

Name:__________________________

Section I:
In this section, there are 12 questions, fill in the correct answers in the spaces provided at the end of each question. Each correct answer is worth 5 points.

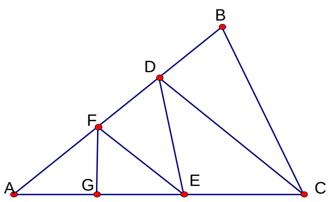
1. Let O_1, O_2 be the centers of circles C_1, C_2 in a plane respectively, and the circles meet at two distinct points A, B. Line O_1A meets the circle C_1 at point P_1, and line O_2A meets the circle C_2 at point P_2. Determine the maximum number of points lying in a circle among these 6 points A, B, O_1, O_2, P_1 and P_2.

Answer:________________________.

2. Suppose that a, b, c are real numbers satisfying $a^2+b^2+c^2=1$ and $a^3+b^3+c^3=1$. Find all possible value(s) of $a+b+c$.

Answer:________________________.

3. In triangle *ABC* as shown in the figure below, *AB*=30, *AC*=32. *D* is a point on *AB*, *E* is a point on *AC*, *F* is a point on *AD* and *G* is a point on *AE*, such that triangles *BCD*, *CDE*, *DEF*, *EFG* and *AFG* have the same area. Find the length of *FD*.



Answer:________________________.

4. The plate number of each truck is a 7-digit number. None of 7 digits starts with zero. Each of the following digits: 0, 1, 2, 3, 5, 6, 7 and 9 can be used only once in a plate, but 6 and 9 cannot both occur in the same plate. The plates are released in ascending order (from smallest number to largest number), and no two plates have the same numbers. So the first two numbers to the last one are listed as follows: 1023567, 1023576, ... , 9753210. What is the plate number of the 7,000 th truck?

Answer:________________________.

5. Determine the number of ordered pairs (x, y) of positive integers satisfying the equation $x^2 + y^2 - 16y = 2004$.

Answer:__________________________pair(s).

6. There are plenty of 2×5 、1×3 small rectangles, it is possible to form new rectangles without overlapping any of these small rectangles. Determine all the ordered pairs (m, n) of positive integers where $2 \le m \le n$, so that no $m\times n$ rectangle will be formed.

Answer:__________________________.

7. Fill nine integers from 1 to 9 into the cells of the following 3×3 table, one number in each cell, so that in the following 6 squares (see figure below) formed by the entries labeled with * in the table, the sum of the 4 entries in each square are all equal.

*		*
*		*

	*	
*		*
	*	

*	*	
*	*	

	*	*
	*	*

*	*	
*	*	

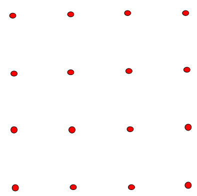
	*	*
	*	*

Answer:

8. A father distributes 83 diamonds to his 5 sons according to the following rules:
(i) no diamond is to be cut;
(ii) no two sons are to receive the same number of diamonds;
(iii) none of the differences between the numbers of diamonds received by any two sons is to be the same;
(iv) Any 3 sons receive more than half of total diamonds.
Give an example how the father distribute the diamonds to his 5 sons.

Answer:__________________________.

9. There are 16 points in a 4×4 grid as shown in the figure. Determine the largest integer n so that for any n points chosen from these 16 points, none 3 of them can form an isosceles triangle.



Answer:___________________________.

10. Given positive integers x and y, both greater than 1, but not necessarily different. The product xy is written on Albert's hat, and the sum $x+y$ is written on Bill's hat. They can not see the numbers on their own hat. Then they take turns to make the statement as follows:

    Bill: " I don't know the number on my hat."
    Albert: " I don't know the number on my hat."
    Bill: "I don't know the number on my hat."
    Albert: "Now, I know the number on my hat."

    Given both of them are smart guys and won't lie, determine the numbers written on their hats.

Answer: Albert's number =___________, Bill's number =_____________.

11. Find all real number(s) x satisfying the equation $\{(x+1)^3\} = x^3$, where $\{y\}$ denotes the fractional part of y, for example $\{3.1416\ldots\} = 0.1416\ldots$.

Answer:___________________________.

12. Determine the minimum value of the expression

    $x^2 + y^2 + 5z^2 - xy - 3yz - xz + 3x - 4y + 7z$,

    where x, y and z are real numbers.

Answer:___________________________.

Section II: Answer the following 3 questions, and show your detailed solution in the space provided after each question. Write down the question number in each paper. Each question is worth 20 points.

1. A sequence $\left(x_1, x_2, \cdots, x_m\right)$ of m terms is called an OE-sequence if the following two conditions are satisfied:
   a. for any positive integer $1 \le i \le m-1$, we have $x_i \le x_{i+1}$;
   b. all the odd numbered terms x_1, x_3, x_5, ...are odd integer, and all the even numbered terms x_2, x_4, x_6,... are even integer.

   For instance, there are only 7 OE-sequences in which the largest term is at most 4, namely, (1), (3), (1,2), (1,4), (3, 4), (1, 2, 3) and (1, 2, 3, 4).
   How many OE-sequences are there in which the largest terms are at most 20? Explain your answer.

2. Suppose the lengths of the three sides of ΔABC are 9, 12 and 15 respectively. Divide each side into $n\ (\geq 2)$ segments of equal length, with $n-1$ division points, and let S be the sum of the square of the distances from each of 3 vertices of ΔABC to the $n-1$ division points lying on its opposite side.

If S is an integer, find all possible positive integer *n*, with detailed answers.

3. Let ABC be an acute triangle with $AB=c$, $BC=a$, $CA=b$. If D is a point on the side BC, E and F are the foot of perpendicular from D to the sides AB and AC respectively. Lines BF and CE meet at point P. If AP is perpendicular to BC, find the length of BD in terms of a, b, c, and prove that your answer is correct.